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Sequential Bargaining without a Common Prior on the Recognition Process

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Abstract

We analyze a sequential bargaining model, where players are allowed to hold different beliefs about which player will make an offer and when. Excessive optimism about making offers in the future can cause a delay in agreement. Despite this, the main result states that, if players will remain sufficiently optimistic for a sufficiently long future, then in equilibrium they will agree immediately.

1 Introduction

Considering a strict procedure that determines which player will make an offer and when, Stahl (1972) and Rubinstein (1982) show that, when the delay is costly, in equilibrium, two players will reach an agreement immediately. Nevertheless, when there are no such strict procedures, players may hold any beliefs about the negotiation process, thereby holding any beliefs about what each player will get in case of a delay. In particular, each player may be so optimistic about what he will get in case of a delay that they may not reach an agreement at the beginning.

In this paper, we will analyze the problem of reaching an agreement in a model that extends the Rubinstein-Stahl framework, where we will allow players to hold their own (possibly optimistic) beliefs about *the recognition process* determining which player will make an offer when. We take the lack of a common prior as the only source of the differences in beliefs.[1]

In our model, the recognition process is the only source of bargaining power. In equilibrium, the recognized player at a given date extracts a non-informational rent, as he makes an offer that can be rejected only by destroying some of the pie. These rents constitute the bargaining power: a player's continuation value is the present value of the rents he expects to extract when he is recognized in the future. Therefore, our analysis will help us to explore the broader question of when two rational individuals can reach an agreement even if they hold incompatible beliefs about which parties will bring which advantages to the table if they keep bargaining.

In our model, we have optimism about a date t whenever each player thinks that the probability that he will be recognized (and hence will make an offer) at t is higher than what the other player assesses. We measure the level of optimism about t by $y_t = p_t^1 + p_t^2 - 1$, where p_t^i is the probability player i assigns to the event that i is recognized at t.

Excessive optimism can cause a delay. To see this, consider the case where two risk-neutral players are trying to divide a dollar, which is worth 1 at $t = 0$, $\delta \in (1/2, 1)$ at $t = 1$, and zero afterwards. It is also common knowledge that each player believes with probability 1 that he will be recognized (and hence will make an offer) at $t = 1$ independent of recognition at $t = 0$. Since the dollar is worth zero afterwards, at $t = 1$, each player is willing to accept any division, hence the recognized player takes the whole dollar. Anticipating this, at $t = 0$, each player expects to take the whole dollar next day, which is worth δ. Thus, they can agree on a division at $t = 0$ only if each gets at least δ, requiring a minimum total amount of $2\delta > 1$. Since they have only one dollar, they cannot reach an agreement at $t = 0$. Therefore, in equilibrium, it is common knowledge at the beginning that players will not reach an agreement before $t = 1$.

[1] If players' beliefs are common knowledge and the common-prior assumption holds, then they must hold the same beliefs, as shown by Aumann (1976). Hence, the differences between players' beliefs may be caused either by lack of common knowledge (as in Myerson (1979) and Kennan and Wilson, 1993), or by lack of a common prior (as in Hicks (1932), Landes (1971), Gould (1973), Posner (1974), and Farber and Bazerman, 1989). For plausibility of the common prior assumption, see Kreps (1990), Aumann (1998) and Gul (1998).

In contrast with this Example, our main result is an Agreement Theorem. It states that, if the players will remain *sufficiently optimistic* for a sufficiently long while, in equilibrium, they will reach an agreement *immediately*. That is, the excessive optimism alone cannot be a reason for a delay in agreement. The reason is simple: A player who is optimistic about the future will not settle for a small share, even if his opponent believes that he would get even less if they wait. Hence, when his opponent recognizes that he will remain very optimistic for a very long while, she will lower her expectations. In equilibrium, their expectations will be so low that they will agree immediately.

These two seemingly conflicting results share a common intuition. As in our Example, players will delay the agreement if they are very optimistic about getting a very high rent in the near future. But the size of this rent depends upon players' expectations in the future. Particularly, if their expectations about the rent at $t+1$ are high, the rent at t will be small. If the players will remain sufficiently optimistic for a sufficiently long while, in equilibrium, the rents in the near future will be so small that each player will prefer to agree on his opponents terms rather than waiting and getting these rents. Hence our Agreement Theorem.

As our Example indicates, it is crucial for our Agreement Theorem that the optimism stays high for a long while. Generalizing this example, in Subsection 3.4 we show that, if players are excessively optimistic for some t^* in the near future, while they are excessively pessimistic for t^*+1 (i.e., if y_{t^*} is very high while y_{t^*+1} is very low), then the agreement may be delayed until t^*. Here, t^* need not be very small: the delay can be so long that almost half the gains from trade is lost.[2]

In order to cause such a long delay, the change in the level of optimism must also be abrupt. For the special case of transferrable utility and independently-distributed recognition process, Theorem 2 states that they will reach an agreement immediately, so long as the level of optimism does not drop substantially at any given date (i.e., so long as $y_t - y_{t+1}$ is sufficiently low at each t), relaxing the hypothesis of our Agreement Theorem in this case.

Theorem 2 identifies the assumption behind the immediate agreement result in the Rubinstein-Stahl framework. It is that the level of optimism stays constant – not necessarily that the optimism is absent.

These results also shed some light on the *deadline effect* observed in some bargaining experiments (Roth et al, 1988) and in some real-world negotiations, where players commonly settle just before the deadline. Above, t^* may be a deadline, after which players do not hope to make any offer. Hence, if the players are very optimistic about making the final offer at t^*, by Subsection 3.4, players will settle just before the deadline, as it is observed. On the other hand, Theorem 2 suggests a further test for our theory: when the deadline is random

[2]This is the theoretical bound of the length of deterministic delays. When people learn from the past, the agreement depends upon the history, in which case the length of the delay can be unbounded (Example 1).

and players can make offers very fast, the deadline effect must disappear.

By requiring the level of optimism to stay high for a long while, our Agreement Theorem implicitly requires that players' beliefs are sufficiently firm. That is, they do not update their beliefs drastically as they observe who gets a chance to make an offer and when. (We will call such change of beliefs "learning.") If a player i is known to update his beliefs fast, the other player, say j, may believe that, if they disagree now, in a short while i will realize that he is wrong, and thus be willing to agree on j's terms. In that case, j's expectation about the future will be higher than what it would be if i's beliefs were very firm. But i does not find it very likely that he will change his mind; he might even expect that j will change her mind. This may cause a disagreement.

Furthermore, in a typical learning process, agents learn faster at the beginning, in which case the disagreement will be at the beginning of the game, and hence we will observe a delay in agreement on the path of equilibrium. Considering a canonical model of learning and confining ourself to the transferrable-utility case, we show for the infinite-horizon case that there exists some date t^{**} such that players will never reach an agreement before t^{**}, and they will always reach an agreement thereafter. We further demonstrate that, when players' initial beliefs are not firm, t^{**} can be large, generating long delays in equilibrium. It is common knowledge at the beginning of the game that they will disagree until t^{**}, distinguishing our theorem from usual delay results in bargaining with private information.

The outline of the paper is as follows. In the next section, we lay out or model. In Section 3, we analyze the special case of transferrable utility and independently-distributed recognition process. There, we use only simple algebra to derive our main results. We describe the equilibria in Section 4, and present our Agreement Theorem in Section 5. Section 6 contains the analysis of our canonical model of learning. After discussing the impact of risk-aversion on reaching an agreement in Section 7, we conclude in Section 8.

2 Model

In this section we will lay out our model. We will write $\mathbb{R}^k$ for any k-dimensional Euclidean space, $\mathbb{N}$ for the set of non-negative integers.

We take a grid $T = \{t \in \mathbb{N} | t < \bar{t}\}$ to be our time space for some $\bar{t} \leq \infty$, a set $N = \{1, 2\}$ to be our set of players, and a compact and convex set $U \subset \mathbb{R}^2$ to be the set of all feasible expected utility pairs. Throughout the paper, we assume that U contains 0 and at least another member that is strictly greater than 0. We will further assume the following the following regularity condition.

Assumption FDA Given any $(u^1, u^2) \in U$, any distinct $i, j \in N$, and any $v^i \in [0, u^i)$, there exists some $v^j > u^j$ such that $(v^1, v^2) \in U$.

That is, the frontier of U is decreasing on the non-negative quadrant. This condition is satisfied under the free-disposal assumption with locally insatiable preferences.

We will analyze the following game, denoted by $G^{\bar{t}}[\delta,\rho]$. At each $t \in T$, Nature recognizes a player $i \in N$; i offers an alternative $u = (u^1, u^2) \in U$; if the other player accepts the offer, then the game ends yielding a payoff vector $\delta^t u = (\delta^t u^1, \delta^t u^2)$ for some $\delta \in (0,1)$; otherwise, the game proceeds to date $t+1$, except for $t = \bar{t} - 1$, when the game ends yielding payoff vector $(0,0)$. We write $\rho = \{\rho_t\}_{t \in T}$ for the publicly observed stochastic process that recognizes the players (so that they can make offers), $\boldsymbol{\rho}_t = (\rho_0(\omega), \rho_1(\omega), \ldots, \rho_{t-1}(\omega)) \in N^t$ for a generic history of the recognized players at the beginning of date t, and $P^i(\cdot|\boldsymbol{\rho}_s)$ for an agent i's conditional beliefs at any history $\boldsymbol{\rho}_s \in N^s$. Players recall everything happened in the past; and everything described in this paragraph is common knowledge.

Notice that we have two sets of beliefs, one for each player; this is our only departure from the Rubinstein-Stahl framework.[3]

Given any $i \in N$ and any history $\boldsymbol{\rho}_s \in N^s$, we write

$$p_t^i(\boldsymbol{\rho}_s) = P^i(\rho_t = i|\boldsymbol{\rho}_s)$$

for the probability player i assigns to the event that he is going to be recognized at date $t \geq s$. By a *belief structure*, we mean any full list $p = \{p_t^i(\boldsymbol{\rho}_s)\}_{i,t,\boldsymbol{\rho}_s}$ of such probability assessments. We will also sometimes write $G^{\bar{t}}[\delta, p]$ for the game where the belief structure is p.

Take any date $t \in T$ and any history $\boldsymbol{\rho}_s \in N^s$ with $s \leq t$. Since we excluded the contingency that no player is recognized, players' probability assessments agree at $(t, \boldsymbol{\rho}_s)$ iff $p_t^1(\boldsymbol{\rho}_s) + p_t^2(\boldsymbol{\rho}_s) = 1$. Now, if $p_t^1(\boldsymbol{\rho}_s) + p_t^2(\boldsymbol{\rho}_s) > 1$, then each player thinks that the probability that he is going to be recognized at t is higher than what the other player assesses. As explained in the Introduction, being recognized is not bad; so we say that *players are optimistic for t at $\boldsymbol{\rho}_s$* iff $p_t^1(\boldsymbol{\rho}_s) + p_t^2(\boldsymbol{\rho}_s) \geq 1$. Likewise, we say that *players are pessimistic for t at $\boldsymbol{\rho}_s$* iff $p_t^1(\boldsymbol{\rho}_s) + p_t^2(\boldsymbol{\rho}_s) \leq 1$. We write

$$y_t(\boldsymbol{\rho}_s) = p_t^1(\boldsymbol{\rho}_s) + p_t^2(\boldsymbol{\rho}_s) - 1$$

for the level of optimism for t. Finally, common-prior assumption in our context corresponds to

$$y_t \equiv 0 \qquad (\forall t \in T). \tag{CPA}$$

When CPA holds, by No-trade Theorem of Milgrom and Stokey (1982), players have no incentive to bet on ρ. Since we are particularly interested in the case where CPA fails (i.e.,

[3] With independently distributed ρ, our model is also isomorphic to an extension of the model of Sakovics (1993), originally developed by Perry and Reny (1993), a bargaining model where players endogenously choose when to make an offer.

$y \neq 0$), our players do have an incentive to bet on ρ; and they are precluded from doing so merely by our imbedded assumption that the side-bets are not feasible (e.g., they are not enforceable).[4]

Notation We will designate dates $t, s \in T$, histories $\boldsymbol{\rho}_t \in N^t$, $\boldsymbol{\rho}_s \in N^s$, players $i, j, k \in N$ with $i \neq j$, and a belief structure p as generic members. Note that p and y are functions defined on a large space $T \times (\cup_{s \in T} N^s)$ and, for each $t \in T$, p_t^i and y_t are stochastic processes, whose values at any s and $\boldsymbol{\rho}_s$ are $p_t^i(\boldsymbol{\rho}_s)$ and $y_t(\boldsymbol{\rho}_s)$, respectively.

3 Dividing a Dollar – An Intuitive Exposition

In this section we confine ourself to the simple bargaining problem of dividing a dollar, and analyze the subgame-perfect equilibria of a *finite-horizon* game $G^{\bar{t}}[\delta, \rho]$ under the following two assumptions of transferrable utility and no learning from previous recognitions.

Assumption TU $U = \{u \in [0,1]^2 | u^1 + u^2 \leq 1\}$.

Assumption NL ρ is independently distributed under both probability distributions P^1 and P^2.

Note that, according to Assumption NL, p is deterministic, i.e., $p_t^i(\boldsymbol{\rho}_s) = p_t^i(\boldsymbol{\rho}'_{s'})$ at any two histories $\boldsymbol{\rho}_s$ and $\boldsymbol{\rho}'_{s'}$ with $s, s' \leq t$. Hence, $p_t^i \in [0,1]$ and $y_t \in [-1,1]$.

Under these two assumptions, we will first construct the equilibrium, and prove the Agreement Theorem, stating that players will reach an agreement immediately, so long as the game is sufficiently long and $y \geq 0$. Then, allowing y to take both positive and negative values, we will present an example with a long delay, and discuss the general properties of delay in our model. Finally, we will present another version of Agreement Theorem that requires only that y does not drop suddenly.

3.1 Continuation values in Equilibrium

For any given finite-horizon game $G^{\bar{t}}[\delta, p]$, we first construct a subgame-perfect equilibrium; any subgame perfect equilibrium is payoff equivalent to the one we construct.

Let us write V_t^i for the continuation value of player i at the beginning of date t, and $S_t = V_t^1 + V_t^2$ for the "perceived size of the pie." If they have not reached an agreement before

[4] Even if the side-bets on ρ were feasible, agents could make a side-bet on ρ_t void by agreeing prior to t. On the other hand, if there exists another contractible process B that remains publicly observable regardless what players do and that yields ρ_t measurable with respect to $\sigma(B_0, \ldots, B_t)$, then we would simply have a general immediate-agreement theorem similar to the one in Yildiz (1998). For they can simply emulate the side-bets on ρ by some contracts on B.

$\bar{t}$, each player will get 0; hence, we set

$$V_{\bar{t}}^1 = V_{\bar{t}}^2 = S_{\bar{t}} = 0. \tag{1}$$

Taking this as final values, we determine V and S, recursively. To do this, let us take any $t \in T$. There are two cases we need to address separately. We first consider the case $\delta S_{t+1} > 1$. If they do not agree at t, each i will get V_{t+1}^i at $t+1$, which is equivalent to δV_{t+1}^i at t. They will then agree on a utility pair $u = (u^1, u^2)$ only if $u^i \geq \delta V_{t+1}^i$ at each $i \in N$, which requires that $u^1 + u^2 \geq \delta V_{t+1}^1 + \delta V_{t+1}^2 = \delta S_{t+1} > 1$, showing that such u is not feasible. That is, they will not reach an agreement at t. Therefore, the continuation value of each i at the beginning of date t will be:

$$V_t^i = \delta V_{t+1}^i \qquad (\delta S_{t+1} > 1)\,. \tag{2}$$

Adding up both sides of the equation over N, we obtain

$$S_t = \delta S_{t+1} \qquad (\delta S_{t+1} > 1)\,. \tag{3}$$

Note that their beliefs about recognition at t is immaterial to the problem.

Now we consider the case $\delta S_{t+1} \leq 1$. Any player j accepts an offer iff it gives him at least δV_{t+1}^j. If the recognized player i offers to give δV_{t+1}^j to j, he is left with $1 - \delta V_{t+1}^j$, which is now at least as high as his continuation value δV_{t+1}^i. Thus, in equilibrium, player i offers to give δV_{t+1}^j to j and to take $1 - \delta V_{t+1}^j$ to himself; and the offer is accepted. Therefore, continuation value of player i at the beginning of date t is:

$$\begin{aligned} V_t^i &= p_t^i(1 - \delta V_{t+1}^j) + (1 - p_t^i)\delta V_{t+1}^i \\ &= p_t^i(1 - \delta S_{t+1}) + \delta V_{t+1}^i \qquad (\delta S_{t+1} \leq 1)\,. \end{aligned} \tag{4}$$

That is, the continuation value of player i at t is the present value δV_{t+1}^i of his continuation value at $t+1$ plus the rent $p_t^i(1 - \delta S_{t+1})$ he expects to extract when he makes an offer. By adding equation (4) up for players, we obtain

$$S_t = 1 + y_t - \delta y_t S_{t+1} \qquad (\delta S_{t+1} \leq 1). \tag{5}$$

Briefly, the processes V^1, V^2, and S defined by the equations (1-5) describes a subgame-perfect equilibrium. Furthermore, V^1 and V^2 are the only continuation values consistent with any subgame-perfect equilibrium; and S prescribes whether they agree at any subgame-perfect equilibrium. Note that, in equilibrium, agreement at any date is fully determined by the aggregate data y, δ (and S).

3.2 Agreement and Disagreement Regimes

Our equilibrium has two regimes. The first one, which we call *disagreement regime*, is characterized by inequality $S_{t+1} > 1/\delta$. In this regime, if they have not yet reached an agreement, players do not reach an agreement at t, either. Hence, their beliefs about which player will be recognized at t is irrelevant; and the perceived size of the "pie" shrinks geometrically as we go back in time.

If $\hat{t}$ is the first date they reach an agreement after a period of disagreement regimes,[5] the perceived size of the pie at any t in that period will be

$$S_t = \delta^{\hat{t}-t} S_{\hat{t}}. \tag{6}$$

Hence, the length of such a period will be

$$L(S_{\hat{t}}, \delta) = \left\lceil \frac{\log S_{\hat{t}}}{\log(1/\delta)} \right\rceil - 1, \tag{7}$$

where operator $\lceil \cdot \rceil$ finds the smallest integer that is greater than or equal to the argument. Since $S_{\hat{t}}$ can be at most 2, the length of a disagreement period can be at most

$$\bar{L}(\delta) = \left\lceil \frac{\log 2}{\log(1/\delta)} \right\rceil - 1. \tag{8}$$

Hence, given any δ, the lengths of disagreement periods are bounded uniformly. In efficiency terms, the length of delay can be at most as large as almost half the pie is lost, i.e., $1/2 < \delta^{\bar{L}(\delta)} \leq 1/(2\delta)$.

The second regime is called the *agreement regime*, and characterized by inequality $S_{t+1} \leq 1/\delta$. In this regime, if they have not reached an agreement yet, players immediately agree on a division that gives the recognized player (say i) $1 - \delta V^j_{t+1}$, which is higher than δV^i_{t+1}; and this is what he would get if the other player j were recognized. That is, the recognized player extracts a rent

$$R_t = 1 - \delta V^j_{t+1} - \delta V^i_{t+1} = 1 - \delta S_{t+1}.$$

The discrepancy among players' beliefs on which player will get this rent causes the discrepancy between the perceived size S_t of the pie and its actual size of 1. In fact, we can re-write equation (5) as

$$S_t = 1 + y_t(1 - \delta S_{t+1}) = 1 + y_t R_t \tag{9}$$

so that the discrepancy is $S_t - 1 = y_t R_t$. Therefore, a high rent for the recognized player aligned with excessive optimism may prevent players from reaching an agreement in some previous dates.

[5] Since they agree at $\bar{t} - 1$, there is such a date for each disagreement period.

If $y_t \leq 0$ and we have an agreement regime at t, then by (9), we have an agreement regime at $t-1$, as well. Thus, if $y_t \leq 0$ at each $t \in T$, then we will have agreement regimes throughout the game. In fact, this case can be imbedded in the Rubinstein-Stahl framework by recognizing each player i with probability p_t^i, and by recognizing nobody with probability $1 - p_t^1 - p_t^2$. We will now analyze the case of $y \geq 0$.

3.3 Perpetual Optimism

Confining ourselves to long but finite games, in this section, we will show that players will reach an agreement immediately, so long as players are optimistic throughout the game, i.e.,

$$y_t \geq 0 \qquad \forall t \in T. \tag{PO}$$

We start with the following Lemma, giving us an agreement regime at each date $s \leq t$ whenever the pie at $t+1$ is perceived to be of the "right size."

Lemma 1 *Assume TU and NL. Given any t with $y_t \geq 0$, if $S_{t+1} \in [1, 1/\delta]$, then $S_t \in [1, 2-\delta] \subset [1, 1/\delta]$.*

Proof. Assume that $S_{t+1} \in [1, 1/\delta]$. Then, we have an agreement regime at t, and hence by (9), we have $S_t = 1 + y_t R_t$, where the rent $R_t = 1 - \delta S_{t+1} \in [0, 1-\delta]$ is bounded from above. Since $y_t \in [0,1]$, $y_t R_t \in [0, 1-\delta]$ and therefore $S_t = 1 + y_t R_t \in [1, 2-\delta]$. Note that $2 - \delta \leq 1/\delta$ (with equality only at $\delta = 1$). ■

Under perpetual optimism (PO), Lemma 1 gives us immediate agreement for sufficiently long games. To see this, first note that, by (PO), $y_{\bar{t}-1} \geq 0$, hence $S_{\bar{t}-1} = 1 + y_{\bar{t}-1} \geq 1$. When $y_{\bar{t}-1} \leq (1-\delta)/\delta$, we have $S_{\bar{t}-1} = 1 + y_{\bar{t}-1} \leq 1/\delta$ so that $S_{\bar{t}-1} \in [1, 1/\delta]$. In that case, using Lemma 1, we can conclude via mathematical induction that $S_t \in [1, 1/\delta]$ at each $t \leq \bar{t} - 1$, showing that we have an agreement regime at each $t \in T$.

Now we consider the case when $y_{\bar{t}-1} > (1-\delta)/\delta$. In this case, we have $S_{\bar{t}-1} = 1 + y_{\bar{t}-1} > 1/\delta$, and hence a disagreement regime at $\bar{t} - 2$. In fact, we have already shown that there is a period of disagreement regimes with length $L(S_{\bar{t}-1}, \delta) \leq \bar{L}(\delta)$ ending at $\bar{t} - 2$. Now, assuming that the game is sufficiently long, consider the last date with an agreement regime before $\bar{t} - 2$, namely $\tilde{t} = \bar{t} - 2 - L(S_{\bar{t}-1}, \delta)$. By definition, we have $S_{\tilde{t}+1} \leq 1/\delta$ and $S_{\tilde{t}+2} > 1/\delta$. The latter inequality also implies that $S_{\tilde{t}+1} = \delta S_{\tilde{t}+2} > 1$, i.e., $S_{\tilde{t}+1} \in [1, 1/\delta]$. Once again, using Lemma 1, we can conclude via mathematical induction that $S_{t+1} \in [1, 1/\delta]$ at each $t \leq \tilde{t}$, showing that we have an agreement regime at each $t \leq \tilde{t}$. To sum up, in this case, for $\bar{t} \geq L(S_{\bar{t}-1}, \delta) + 2$, we have three periods of constant regimes: We have agreement regimes at $\{0, \ldots, \bar{t} - 2 - L(S_{\bar{t}-1}, \delta)\}$, disagreement regimes at $\{\bar{t} - L(S_{\bar{t}-1}, \delta) - 1, \ldots, \bar{t} - 2\}$, and again an agreement regime at $\bar{t} - 1$. (This is exhibited in Figure 1 for the extreme case when $y_t = p_t^1 = p_t^2 = 1$ at each $t \in T$.)

In either case, we have an agreement regime at each $t \leq \bar{t} - 2 - \bar{L}(\delta) \leq \bar{t} - 2 - L(S_{\bar{t}-1}, \delta)$, proving our first Agreement Theorem:

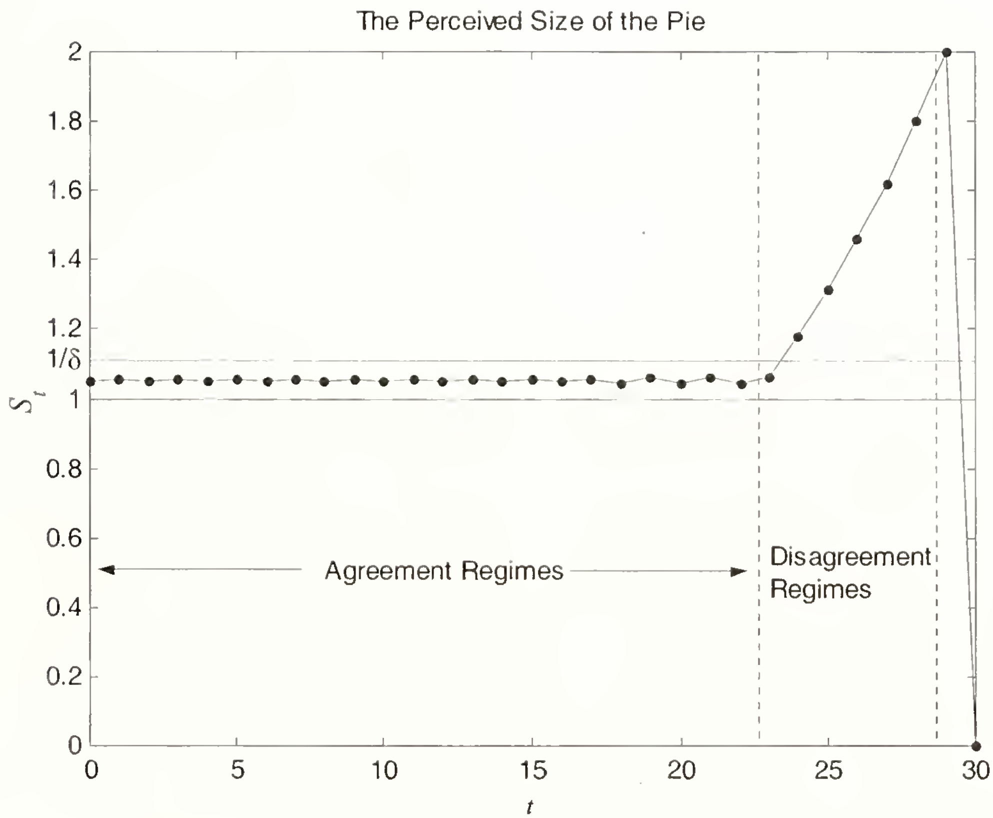

Figure 1: The perceived size of the pie in the extreme case where $y_t = p_t^1 = p_t^2 = 1$ at each $t \in T$. [$\bar{t} = 30$, $\delta = 0.9$.] Note that we have an agreement regime at $\bar{t} - 1$, too.

Theorem 1 *Assume TU, NL, and PO. Then, we have an agreement regime at each* $t \in T$ *with* $t < \bar{t} - \bar{L}(\delta) - 2$.

Excluding the end of the game, our Theorem states that, if players are optimistic throughout the game, they will reach an agreement immediately, suggesting that the Example we have given in the Introduction may be rather singular. Actually, our Example illustrated that sequential bargaining allocates a non-negative rent for the recognized player at any date t; and this rent can be very large. If players are also extremely optimistic about getting this large rent, they may choose to wait at $t-1$.

Our Theorem goes one step beyond this intuition. At such an agreement regime t, if players are also sufficiently optimistic for subsequent times, the rent for the recognized player will be small, a fact that we established in Lemma 1. Our Theorem states that, in equilibrium, this rent will be so small that the players will reach an agreement at $t-1$ even if they are very optimistic about getting it.

3.3.1 Perpetual Optimism – Infinite Horizon

Theorem 1 establishes that, in case of perpetual optimism, we have agreement regimes throughout the game except for a bounded period just before the end. Naturally, as $\bar{t} \to \infty$, this period of disagreement regimes disappears, and thus we have agreement regimes throughout the game – like in the Rubinstein-Stahl framework.

Take any $\delta \in (0,1)$ and any infinite belief structure p. For any sufficiently long truncation $G^{\bar{t}}$, write $S[\bar{t}]$ and $\tilde{t}[\bar{t}]$ for the size of the pie and the last date before any disagreement, respectively. Given any $t < \tilde{t}[\bar{t}]$, we have an agreement regime at each s with $t \le s \le \tilde{t}[\bar{t}]$; hence, solving (5) on this interval, we obtain

$$S_t[\bar{t}] = \sum_{s=t}^{\tilde{t}[\bar{t}]-1} \pi_t^s (1 + y_s) + \pi_t^{\tilde{t}[\bar{t}]} S_{\tilde{t}[\bar{t}]},$$

where π_t^s is $\prod_{j=t}^{s-1}(-\delta y_j)$ when $t < s$ and 1 when $t = s$. As $\bar{t} \to \infty$, $\tilde{t}[\bar{t}] \to \infty$, hence $\pi_t^{\tilde{t}[\bar{t}]} \to 0$; and since $0 \le S_{\tilde{t}[\bar{t}]} \le 2$, $S_t[\bar{t}]$ converges to

$$S_t[\infty] = \sum_{s=t}^{\infty} \pi_t^s (1 + y_s). \tag{10}$$

By Theorem 1, $S_t[\bar{t}] \in [1, 1/\delta]$ for every sufficiently large $\bar{t}$ so that $S_t[\infty] \in [1, 1/\delta]$ at each $t \in \mathbb{N}$. That is, for $\bar{t} = \infty$, we have agreement regimes throughout: at any t, the recognized player i gives the other player, say j, $\delta V_{t+1}^j[\infty] \in [0, \delta]$, keeping $1 - \delta V_{t+1}^j[\infty] \in [1-\delta, 1]$ for himself – a behavior that replicates the equilibrium behavior in the Rubinstein-Stahl framework, characterized by CPA. In fact, when CPA holds (i.e., $y \equiv 0$), $V_t^i[\infty]$ can take any value

in $[0, 1]$, thus any such division is consistent with CPA; and therefore CPA (and thereby the Rubinstein-Stahl framework) is not refutable in the case of perpetual optimism.

To see this, let $y_t = \bar{y} \in [0, 1]$ at each $t \in \mathbb{N}$. Now, by inserting $\bar{y}$ for each y_s in (10), one can easily compute that $S_t[\infty] = \frac{1+\bar{y}}{1+\delta\bar{y}}$, and hence $R_t[\infty] = 1 - \delta S_{t+1}[\infty] = \frac{1-\delta}{1+\delta\bar{y}} \equiv \bar{R}$. Moreover, by Proposition 2 of Section 4, we have $V_t^i[\infty] = \sum_{s=t}^{\infty} \delta^{s-t} p_s^i R_s[\infty]$. Thus, $V_t^i[\infty] = \bar{R} \sum_{s=t}^{\infty} \delta^{s-t} p_s^i$. Since $p_s^i \in [\bar{y}, 1]$ at each $s \in T$, the range of $(1-\delta) \sum_{s=t}^{\infty} \delta^{s-t} p_s^i$ is $[\bar{y}, 1]$, and therefore the range of $V_t^i[\infty]$ is $\left[\frac{\bar{y}}{1+\delta\bar{y}}, \frac{1}{1+\delta\bar{y}}\right]$. When CPA holds (i.e., at $\bar{y} = 0$), this interval is entire $[0, 1]$, thus any outcome is consistent with CPA.

As $\bar{y}$ increases, the interval shrinks; and at $\bar{y} = 1$ it consists of $1/(1+\delta)$. That is, in the equilibrium of the extreme case when everyone keeps believing that he will make all the remaining offers no matter what happened so far (i.e., $p_t^1 = p_t^2 = \bar{y} = 1 \; \forall t \in T$), the recognized player offers to take $1/(1+\delta)$, leaving the other player $\delta/(1+\delta)$; and the offer is accepted. This is the equilibrium outcome under the Alternating-Offer Procedure, analyzed by Rubinstein (1982). Note that Rubinstein's well-known division is the only division that is not ruled out by any level of optimism.

3.4 A Long Delay in Agreement

Theorem 1 establishes immediate agreement for the long games where players remain optimistic. Now, allowing players be pessimistic sometimes, we will create a simple example where the agreement is delayed for a long while, no matter how long the game is. The rationale behind this example is simple. When each player believes that he will not be able to make an offer after a given date, they will act as if the negotiation stops there, imitating a short game.

Since the length of a period of disagreement regimes cannot be longer than $\bar{L}(\delta)$, we already have an upper bound for the delay in agreement. Note that this upper bound is rather long: $\delta^{\bar{L}(\delta)+1} \leq 1/2$, i.e., almost half the benefit is lost. Nevertheless, it is not loose, as our example reveals:

Consider any belief structure p with $y_{t^*} = 1$ and $y_t = -1$ at each $t > t^*$, where t^* is any date with $0 < t^* \leq \min\{\bar{L}(\delta), \bar{t}\}$. At t^* and thereafter, thinking that he will no longer have any other chance to make an offer, each player is willing to accept any offer. Hence, the recognized player can extract a huge rent: we have $R_t = 1$ at each $t \geq t^*$. They consider t^* an auspicious moment, each thinking that he is going to make an offer at t^*. Amplified by the extreme pessimism for the future, their optimism for t^* leads them to perceive the pie at t^* very large:

$$S_{t^*} = 1 + y_{t^*} R_{t^*} = 2 > 1/\delta.$$

The pie at t^* is perceived to be so large that they will never reach an agreement before t^*:

$$S_{t+1} = \delta^{t^*-t-1} S_{t^*} = 2\delta^{t^*-t-1} > 1/\delta, \qquad (0 \leq t < t^*)$$

where the last inequality is due to the definition of $\bar{L}(\delta)$, which is defined by $2\delta^{\bar{L}(\delta)} \in (1, 1/\delta]$. That is, during $\{0, \ldots, t^* - 1\}$, without agreeing, players wait for the auspicious moment t^*, when all the offers are going to be accepted.

This example illustrates some key properties of delays caused by lack of a common prior. Firstly, *the delay in our model is not Pareto-optimal in general.* To see this, consider the case $y_1 = 1$. Now, giving the dollar to the player recognized at $t = 1$ Pareto-dominates the equilibrium outcome, which gives the dollar to the player recognized at t^*. Note that a player's expected utility levels under these two schemes are δ and δ^{t^*}, respectively. [This also shows that the equilibrium outcome in the example presented in the Introduction was Pareto-optimal.]

Second, *at the beginning of our game it is common knowledge that players will not reach an agreement before* t^*; whereas the delay is only a possibility in usual bargaining models with private information, as the types with the least advantageous information reach an agreement immediately in any separating equilibrium in these models.

Third, *there is a consensus among our players that the delay is costly and that there is a division at* $t = 0$ *that gives each player more than what he gets in equilibrium.* That is, they agree that the size of the pie at $t = 1$ is only δ; and each player further thinks that they both would get more than their equilibrium consumption if they divided the dollar at $t = 0$ by giving $\delta + (1 - \delta)/2$ to himself and leaving $(1 - \delta)/2$ to his opponent. There is no consensus, however, on *which* division at $t = 0$ dominates the equilibrium outcome.

Finally, even though there is a consensus that the delay is costly, one cannot find a mechanism that would yield an agreement at the beginning and would be accepted by each player to replace the current situation. In particular, they cannot agree on a procedure that recognizes them with "objective" probabilities throughout the negotiation. For, if such a mechanism were accepted by player 1, it would give him at least δ^*, thus would give player 2 at most $1 - \delta^*$, and would be rejected by player 2.

A period of low y does not necessarily mean that they are unreasonably pessimistic; it may simply reflect the fact that agents are not going to negotiate during this period, as happens in bargaining with a deadline or several "bargaining sessions." In that case, our analysis illustrates that, if the end of the session or the deadline is sufficiently close, agents will wait until the deadline to settle. This *deadline effect* is observed in laboratory experiments and the real-world negotiations.

3.5 A sudden loss of optimism

In the previous section, we illustrated that a sudden loss of optimism may give players an incentive to wait. Now we will establish that, whenever there is a period of disagreement regimes, it must be followed by a sudden loss of optimism.

Towards this goal, our first lemma states that if a disagreement regime proceeds an agree-

ment regime, there must be a substantial drop in y.

Lemma 2 *Given any $G^{\bar{t}}[\delta,p]$ and any $t<\bar{t}-1$, if $S_t>\frac{1}{\delta}$ and $S_{t+1}\leq\frac{1}{\delta}$, then $y_t-y_{t+1}>(1-\delta)/\delta$.*

Proof. Take any $t<\bar{t}-1$ with $S_t>\frac{1}{\delta}$ and $S_{t+1}\leq\frac{1}{\delta}$. Since $S_{t+1}\leq\frac{1}{\delta}$, we have an agreement regime at t, and hence

$$S_t=1+y_t(1-\delta S_{t+1})>\frac{1}{\delta}. \tag{11}$$

Since $1-\delta S_{t+1}\geq 0$, we then have $y_t>0$. Since $S_t>1/\delta$ and $y_t>0$, by Lemma 1, we have $S_{t+1}<1$. In that case, we have $S_{t+2}\leq 1/\delta$, for otherwise we would have a disagreement regime at $t+1$, rendering $S_{t+1}=\delta S_{t+2}>1$. Hence, we have an agreement regime at $t+1$, and thus

$$S_{t+1}=1+y_{t+1}(1-\delta S_{t+2})<1, \tag{12}$$

yielding $y_{t+1}<0$. By combining (11) and (12), and writing $\Delta=y_t-y_{t+1}$, we obtain

$$\frac{1}{\delta}<S_t=1+y_t-\delta y_t(1+y_t)+\delta y_t\Delta+\delta^2 y_t y_{t+1}S_{t+2}. \tag{13}$$

Since $y_ty_{t+1}S_{t+2}\leq 0$, this gives us $\delta y_t\Delta\geq\delta y_t\Delta+\delta^2y_ty_{t+1}S_{t+2}>\frac{1}{\delta}-1-y_t+\delta y_t(1+y_t)$. Hence,

$$\Delta>\frac{1}{\delta y_t}\left(\frac{1}{\delta}-1\right)-\left(\frac{1}{\delta}-1\right)+y_t. \tag{14}$$

Once can check that the expression on the right hand side is minimized at $y_t^*=\frac{1}{\sqrt{\delta}}\sqrt{\frac{1}{\delta}-1}$, taking the value of $2\sqrt{\frac{1}{\delta}\left(\frac{1}{\delta}-1\right)}-\left(\frac{1}{\delta}-1\right)$, which is greater than $1/\delta-1$. ∎

By Lemma 2, if we have a disagreement regime at $t-1$ while we have an agreement regime at t, we must have $y_t-y_{t+1}>(1-\delta)/\delta$. Put it differently, if $y_t-y_{t+1}\leq(1-\delta)/\delta$ and we have an agreement regime at $t<\bar{t}-1$, we must also have an agreement regime at $t-1$. We already know that we have an agreement regime at some $\tilde{t}$ with $\bar{t}-2-\bar{L}(\delta)\leq\tilde{t}<\bar{t}-1$. Hence, if y is smooth enough so that $y_t-y_{t+1}\leq(1-\delta)/\delta$ at each $t\in T$, we would have an agreement regime at each $t\leq\tilde{t}$:

Theorem 2 *Assume TU, NL, and that $y_t-y_{t+1}\leq(1-\delta)/\delta$ at each $t\in T$. Then, we have an agreement regime at each $t\in T$ with $t\leq\bar{t}-\bar{L}(\delta)-2$.*

Excluding the end of the game, Theorem 1 stated that, if players are optimistic throughout the game, they will reach an agreement immediately. Proposition 2 establishes that, actually, we do not need optimism stay high; all we need is that it does not drop too fast. In fact, by

requiring perpetual optimism, Theorem 1 also bounded the drop in y: if $y_t, y_{t+1} \in [0,1]$, then $y_t - y_{t+1} \leq 1$. Of course, this bound is much looser than that of Theorem 2; and these two results are logically independent. We may have a disagreement regime just before the end of the game, precisely because after the game ends y_t is identically -1, and thus we may have a substantial drop in y at the end of the game. Theorem 2 also implies that if the transition to -1 is smooth enough, there will be no disagreement regime at the end of the game, either.

We will present our Agreement Theorem for the general case in Section 5. Beforehand, we present some preliminary results.

4 Equilibrium and the Rents – General case

In this section, we will describe the subgame-perfect equilibria of our game. In the finite-horizon case, all the equilibria will be payoff equivalent to each other. In the infinite horizon case, we will only consider the limit of the equilibria of the finite-horizon truncations. In equilibrium, at each date, the recognized player extracts a non-informational rent, and these rents determine players' equilibrium-payoffs.

Towards describing the subgame perfect equilibria, let us take any finite-horizon game $G^{\bar{t}}[\delta,\rho]$, and write $V_t^i[\bar{t}](\rho_t)$ for the equilibrium continuation-value of a player i at any t with history ρ_t. The stochastic process $V[\bar{t}]$ is defined by the recursive equation

$$V_t^i[\bar{t}](\rho_t) = P^i(\rho_t = i|\rho_t)\, m^i(\delta V_{t+1}[\bar{t}](\rho_t, i)) + P^i(\rho_t = j|\rho_t)\, \delta V_{t+1}^i[\bar{t}](\rho_t, j) \tag{15}$$

and the boundary condition

$$V_{\bar{t}}[\bar{t}] \equiv 0, \tag{16}$$

where $m^i : \mathbb{R}^2 \to \mathbb{R}_+$ is defined through $m^i(v) = \max\left[\{v^i\} \cup \{u^i | \left(u^1, u^2\right) \in U,\ u^j \geq v^j\}\right]$ at each $v \in \mathbb{R}^2$ for $i \neq j$. Note that $m^i(v)$ is the maximum payoff player i can enjoy if he needs the consent of player j for an agreement and if the payoff vector v would be realized in case of a disagreement. (Of course, he will choose to disagree if it is too costly to convince the other player, i.e., when $v \notin U$.) By Assumption FDA, when i gets $m^i(v)$, j gets v^j. Hence, (15) expresses that, with probability $P^i(\rho_t = i|\rho_t)$, player i will be recognized, in which case he will get the maximum payoff given that he needs to give at least $\delta V_{t+1}^j[\bar{t}](\rho_t, i)$ to player j in order to reach agreement, and with probability $P^i(\rho_t = j|\rho_t)$, j will be recognized, in which case i will get his continuation value in the case of a delay. Condition (16) simply states the fact that players automatically get 0 at $\bar{t}$.

The equilibrium strategy $\bar{s}_i[\bar{t}]$ of a player i $(\neq j)$ will be as follows:

> Given any t with ρ_t, if i is recognized at t, and if $\delta V_{t+1}[\bar{t}](\rho_t, i) \in U$, offer to give $\delta V_{t+1}^j[\bar{t}](\rho_t, i)$ to j and $m^i(\delta V_{t+1}[\bar{t}](\rho_t, i))$ to i; if i is recognized, but

$\delta V_{t+1}[\bar{t}](\rho_t, i) \notin U$, then offer to give 0 to j and $m^i(0)$ to i; if j is recognized, accept his offer iff he gives i at least $\delta V^i_{t+1}[\bar{t}](\rho_t, j)$.

Given any infinite-horizon game $G^{\infty}[\delta, \rho]$, for each $\bar{t} \in \mathbb{N}$, consider the finite-horizon truncation $G^{\bar{t}}[\delta, \rho]$, which agrees with $G^{\infty}[\delta, \rho]$ until $\bar{t}$, and ends there. Defining $V[\bar{t}]$ as above, assume that

$$V_t[\bar{t}](\rho_t) \to V_t[\infty](\rho_t) \quad \text{as } \bar{t} \to \infty \qquad (\forall \rho_t, \forall t) \tag{17}$$

for some $V[\infty]$. We define the strategy profile $\bar{s}[\infty] = (\bar{s}_1[\infty], \bar{s}_2[\infty])$ as above, by taking $\bar{t} = \infty$. The following Proposition states that, when $\bar{t}$ is finite, $\bar{s}[\bar{t}] = (\bar{s}_1[\bar{t}], \bar{s}_2[\bar{t}])$ is essentially the unique subgame perfect equilibrium; when $\bar{t} = \infty$, $\bar{s}[\infty]$ is a subgame-perfect equilibrium, but not necessarily unique. The proof of this Proposition is fairly straight-forward, and can be found in Yildiz (2000).

Proposition 1 *Given any $G^{\infty}[\delta, \rho]$ with (17), $\bar{s}[\infty]$ is a subgame-perfect equilibrium. Given any finite-horizon game $G^{\bar{t}}[\delta, \rho]$, $\bar{s}[\bar{t}]$ is a subgame-perfect equilibrium. Moreover, for $\bar{t} < \infty$, at any subgame-perfect equilibrium, and at any date t with ρ_t, the following are true: the vector of continuation values at the beginning of t is $V_t(\rho_t)$; if $\delta V_{t+1}(\rho_t, i)$ is in the interior of U, and i is recognized, then they reach an agreement that gives $\delta V^j_{t+1}(\rho_t, i)$ to j and $m^i(\delta V_{t+1}[\bar{t}](\rho_t, i))$ to i; and if $\delta V_{t+1}(\rho_t, i) \notin U$ and i is recognized, then they do not reach an agreement at t.*

In the rest of the paper, we will confine ourself to the equilibrium $\bar{s}$; we suppress the terms in square brackets.

In equilibrium, the recognized player extract a rent; and these rents constitutes the bargaining power. To see this, observe that, given any ρ_t and any player i, the difference between his payoff at (ρ_t, i), when he makes an offer, and his payoff at (ρ_t, j), when he does not, is $m^i(\delta V_{t+1}(\rho_t, i)) - \delta V^i_{t+1}(\rho_t, j) = \left[m^i(\delta V_{t+1}(\rho_t, i)) - \delta V^i_{t+1}(\rho_t, i)\right] + \left[\delta V^i_{t+1}(\rho_t, i) - \delta V^i_{t+1}(\rho_t, j)\right]$. Here, the last term is informational; that is, the identity of the recognized player at t affects players' beliefs about the future recognitions, which in turn affect their continuation values in equilibrium. The first term,

$$R^i_t(\rho_t) = m^i(\delta V_{t+1}(\rho_t, i)) - \delta V^i_{t+1}(\rho_t, i), \tag{18}$$

on the other hand, is caused by his opportunity to make an offer that can be rejected only by delaying the agreement until $t+1$; and we call it *the (non-informational) rent for i at t*.

Substituting (18) in Equation (15) and carrying out the necessary algebra, we obtain

$$V^i_t(\rho_t) = E^i[1_{\{\rho_t = i\}} R^i_t | \rho_t] + \delta E^i[V^i_{t+1} | \rho_t], \tag{19}$$

where $1_{\{\rho_t = i\}}$ is the indicator function of $\{\rho_t = i\}$, taking values 1 and 0 when $\rho_t = i$ and $\rho_t \neq i$, respectively. Equation (19) states that the value of the game for a player i at any date

t is his discounted expected value of the game at $t+1$ plus the rent he expects to extract from being a recognized player at t. Since the value of the game at $\bar{t}$ is identically nil, this gives us the following Proposition, stating that, in equilibrium, the continuation value of a player is the present value of all the rents he expects to extract in the future. In an agreement, the recognized player gives the other player his continuation value, keeping the rest of the pie for himself. Therefore, a *player's share* in an agreement is *itself* the discounted sum of the rents he expects to extract in the future, plus the current rent in case he is the recognized player.

Proposition 2 *Given any $i \in N$, $t \in T$ and any $\boldsymbol{\rho}_t$, we have*

$$V_t^i(\boldsymbol{\rho}_t) = \sum_{s=t}^{\bar{t}-1} \delta^{s-t} E^i \left[1_{\{\rho_s = i\}} R_s^i | \boldsymbol{\rho}_t \right]. \tag{20}$$

Using (16), (19), and the law of iterated expectations, one can easily check via mathematical induction that (20) holds for any finite $\bar{t}$. Using the fact that R is uniformly bounded, Yildiz (2000) shows that it holds for the infinite-horizon case as well.

Proposition 2 states that the continuation value of a player is the present value of the rents he expects to extract in the future. Since the rents themselves depend on players' beliefs as well as the history, it is not clear that a player would not lose if each player comes to believe that he is more likely to make offers in the future. In fact, in Appendix A we present an example where a player loses as he finds himself more likely to make an offer at $t = 1$, while everything else remains unchanged. Nevertheless, there we also show that a player will not lose when each player comes to believe that he is more likely to make offers in the future, so long as the recognition process is affiliated, e.g., it is exchangeable or independently distributed – the cases that we mainly focus. In that case, our definition of optimism will be unambiguously valid.

According to $\bar{s}$, players reach an agreement at any $(\boldsymbol{\rho}_t, i)$ iff $\delta V_{t+1}(\boldsymbol{\rho}_t, i) \in U$. In that case, we will say that we have an agreement regime at $(\boldsymbol{\rho}_t, i)$, and a disagreement regime, otherwise.

5 Agreement Theorem

Assuming that players do not change their beliefs as they observe who gets a chance to make an offer and when (NL), for the case of transferrable utility (TU), Theorem 1 established that, if the level of optimism y stays *non-negative* for a sufficiently long future, players will reach an agreement immediately. In this section, we will present an extension of this result. Our extension will state that, if the level of optimism y stays *sufficiently high* for a sufficiently long future, players will reach an agreement immediately. While Assumption TU is dropped entirely, it will be clear that a weaker form of NL remains embedded in the assumption that y stays sufficiently high for a sufficiently long future.

We first present the result for the finite horizon case.

Theorem 3 *Given any $\delta \in (0,1)$ and any non-negative integer $t^* \in \mathbb{N}$, there exists some $y^* \in (-1,1)$ such that, for every finite-horizon game $G^{\bar{t}}[\delta,\rho]$ with $\bar{t} \geq t^* + \bar{L}(\delta) + 3$ and with $y_t \geq y^*$ at each $t \in T$, we have an agreement regime at every $t \leq t^*$.*

Theorem 3 extends Theorem 1 in a weaker form, by dropping Assumptions TU and NL. Intuitively, if players know that their opponents will remain *sufficiently optimistic*, they will adjust their expectations down so that they will reach an agreement immediately, except for the end of the game, where we may have a period of disagreement regimes due to a large last-mover-advantage. Under Assumptions NL and TU, Theorem 1 required only that players do not lose their optimism, hence $y^* = 0$ was sufficient for immediate agreement. In Example 3 we will establish that y^* may need to be strictly positive with risk-averse players.

The proof of this Theorem is similar to that of Theorem 1, but less transparent, and therefore relegated to Appendix B. The proof consists of the following steps. Firstly, we show that, under NL, the lengths of disagreement periods are uniformly bounded. Then, we show that our result holds with strict inequalities for the case when y is identically 1. Finally, invoking a continuity property of equilibrium payoffs with respect to the belief structures, we obtain our result.

We extend Theorem 3 to the infinite-horizon case as follows.

Theorem 4 *Given any $\delta \in (0,1)$ and any non-negative integer $t^* \in \mathbb{N}$, assume that*

$$V_t\left[\bar{t}, p\right] \to V_t\left[\infty, p\right] \quad \text{as } \bar{t} \to \infty \qquad (\forall t \leq t^*) \tag{21}$$

uniformly over belief structures p, where $\bar{t}$ and p indicates the underlying game. Then, there exist some $\hat{t} \in \mathbb{N}$ and $y^* \in (-1,1)$ such that, given any $G^\infty[\delta,\rho]$ with $y_t \geq y^*$ at each $t \leq \hat{t}$, we have an agreement regime at every $t \leq t^*$.

Assuming that equilibrium continuation-values converge uniformly over belief structures, Theorem 4 extends Theorem 3 to infinite-horizon games, stating that, if the level of optimism stays sufficiently high for a sufficiently long future, players will reach an agreement immediately. Intuitively, under (21), the infinite horizon can be approximated by a finite horizon, whence a version of Theorem 3 (with strict inequalities) would give us immediate agreement for the infinite horizon case.

Here, a *finite* period of a sufficiently high level of optimism suffices for an immediate agreement. By requiring the optimism stay high only for a finite time, we allow players' beliefs to merge, a plausible property exhibited by many learning models, such as the one we will employ in Section 6.

While Assumption TU is dropped entirely, a weaker form of NL remains embedded in Theorems 3 and 4. By requiring that $y_t \geq y^*$, our Theorem also requires that $p_t^i \geq y^*$ for each i, as $p_t^i = y_t + 1 - p_t^j \geq y_t$. Since $p_t^i \leq 1$, this further requires that $\left|p_t^i(\rho_t) - p_t^i(\rho_t')\right| \leq 1 - y^*$ for any two histories ρ_t and ρ_t', bounding how much can be learned before t.

5.1 Discussion

If a player i believes that his opponent j is very optimistic and will remain so, he will lower his expectations. For such an optimistic opponent will not settle for little, even if i thinks that the value of continuing on bargaining for j is even lower. If the level of optimism will remain sufficiently high for a sufficiently long future, then players' expectation will be so low that they will reach an agreement immediately.

If the level of optimism stays very high for a long while, it must be the case that players' conditional beliefs do not vary much with respect to what happens so far. Since they are Bayesian, this means that they adhere to their initial beliefs, and thus their prior beliefs are very firm. On the other hand, if players' prior beliefs reflect a high level of optimism for a long future and their prior beliefs are firm, then the level of optimism will remain high. Therefore, we can restate our theorem as follows. *If players' prior beliefs reflect a sufficiently high level of optimism for a sufficiently long immediate-future, and if these beliefs are sufficiently firm, then they will reach an agreement immediately.*[6]

Our theorem asks for three conditions for an immediate agreement. Firstly, there must be a sufficiently long future. If the game is to end very soon, and if players are very optimistic about making an offer at the end of the game, each may choose to wait until the end, when his opponent will know that she will get 0 if they disagree – as we demonstrated in the Introduction.

Second, they must remain sufficiently optimistic for a long while. For, even if the game is to continue indefinitely, if each player thinks that he will never be able to make an offer after a given date, they will act as if the game ends there, hence they may disagree at the beginning, as they may do in a short game.[7]

Finally, it asks players' initial beliefs be sufficiently firm, that is, they should not update their beliefs drastically with a limited observation. There are three reasons for this requirement. Firstly, it is essential for our theorems that the length of a period of disagreement regimes is uniformly bounded under NL, as there may be a disagreement period at the end of the game. When players update their beliefs as they observe which players are recognized and when, whether they reach an agreement at a given date depends on which players have been

[6] For formal statements, see Yildiz (2000).

[7] Since players will wait for the end only when they are optimistic about the end, we may only need that the level of optimism does not drop suddenly and drastically, as suggested by Theorem 2. It remains an open question whether we can (qualitatively) extend Theorem 2 beyond Assumptions TU and NL, obtaining a stronger theorem.

recognized and when. The following example demonstrates that, when the beliefs are updated drastically, the length of such a disagreement period may be unbounded, causing (possibly long) delays.

Example 1 *Consider the following belief structure: For each $i \in N$, and for some $\bar{p} > 1/2$, we have $p_t^i(\boldsymbol{\rho}_t) = \bar{p}$ at each "alternating" history $\boldsymbol{\rho}_t = (\rho_0(\omega), \ldots, \rho_{t-1}(\omega))$ where $\rho_s(\omega) \neq \rho_{s+1}(\omega)$ at each $s < t-1$. On the other hand, if a player i has been recognized twice in a row before the other agent, each player thinks that i is "blessed" so that he will be recognized with probability $\bar{p}$, i.e., we have $P^1(\rho_t = \rho_{s^*}(\omega)|\boldsymbol{\rho}_t) = P^2(\rho_t = \rho_{s^*}(\omega)|\boldsymbol{\rho}_t) = \bar{p}$ at each $\boldsymbol{\rho}_t = (\rho_0(\omega), \ldots, \rho_{t-1}(\omega))$ with $s^* = \min\{s < t-1 | \rho_s(\omega) = \rho_{s+1}(\omega)\} \geq 0$. Letting $\bar{t} = \infty$, we will demonstrate for high values of δ that players will never reach an agreement at an alternating history. In that case, given any t, at the beginning of the game each player believes that with positive probability they will not reach an agreement before t.*

Consider any alternating history $\boldsymbol{\rho}_t$, *where i is recognized at $t-1$. If i is recognized at t, too, then he will be revealed to be blessed. In that case, by Proposition 2, we will have $V_{t+1}^i(\boldsymbol{\rho}_t, i) = \sum_{s \geq t} \delta^{s-t} \bar{p}(1-\delta) = \bar{p}$ and $V_{t+1}^j(\boldsymbol{\rho}_t, i) = 1 - \bar{p}$ for $j \neq i$. (Note that $R = 1 - \delta$.) Hence, i will offer to take $1 - \delta(1-\bar{p})$, leaving $\delta(1-\bar{p})$ to j, who will accept the offer. Since $p_t^k(\boldsymbol{\rho}_t) = \bar{p}$ for each $k \in N$, we now have*

$$
\begin{aligned}
V_t^i(\boldsymbol{\rho}_t) &= \bar{p}(1 - \delta(1-\bar{p})) + (1-\bar{p})\delta V_{t+1}^i(\boldsymbol{\rho}_t, j) \\
V_t^j(\boldsymbol{\rho}_t) &= \bar{p}\delta V_{t+1}^j(\boldsymbol{\rho}_t, j) + (1-\bar{p})\delta(1-\bar{p}),
\end{aligned}
\tag{22}
$$

where $(\boldsymbol{\rho}_t, j)$ is again an alternating history. The subgames starting at $\boldsymbol{\rho}_t$ and $(\boldsymbol{\rho}_t, j, i)$ are identical so that $V_t(\boldsymbol{\rho}_t) = V_t(\boldsymbol{\rho}_t, j, i)$, hence, at any alternating history, payoffs depend only on the last player recognized. Furthermore, our game is symmetric with respect to the players so that $V_t^i(\boldsymbol{\rho}_t) = V_{t+1}^j(\boldsymbol{\rho}_t, j) \equiv V^L(\bar{p}, \delta)$ and $V_t^j(\boldsymbol{\rho}_t) = V_{t+1}^i(\boldsymbol{\rho}_t, j) \equiv V^F(\bar{p}, \delta)$. Substituting these values into (22), we obtain

$$
\begin{pmatrix} V^L(\bar{p}, \delta) \\ V^F(\bar{p}, \delta) \end{pmatrix} = \frac{1}{1 - \delta^2 \bar{p}(1-\bar{p})} \begin{bmatrix} 1 & \delta(1-\bar{p}) \\ \delta\bar{p} & 1 \end{bmatrix} \begin{pmatrix} \bar{p}(1 - \delta(1-\bar{p})) \\ \delta(1-\bar{p})^2 \end{pmatrix}. \tag{23}
$$

Letting $\delta \to 1$, one can check from (23) that, for every $\bar{p} > 1/2$ there exists some δ sufficiently close to 1 such that $V^L(\bar{p}, \delta) + V^F(\bar{p}, \delta) > 1/\delta$, when we will have a disagreement regime at each alternating history.

Second, when players update their beliefs drastically, we will have another form of the deadline effect, as illustrated in the following example.

Example 2 *Assuming TU, take any t^* with $0 < t^* \leq \min\{\bar{L}(\delta), \bar{t}\}$. Consider the case where $(\rho_0, \rho_1, \ldots, \rho_{t^*})$ are stochastically independent under both P^1 and P^2, $P^i(\rho_{t^*} = i) = 1$ for each*

$i \in N$, and $\rho_t = \rho_{t^*}$ at each $t > t^*$. If a player i is recognized at t^*, the other player j will know that he will no longer be able to make any offer, hence by Proposition 2, we have $V^j_{t^*+1}(\rho_{t^*}, i) = 0$ at each ρ_{t^*}. In that case, we will have $m^i(\delta V_{t+1}(\rho_{t^*}, i)) = 1 - \delta V^j_{t+1}(\rho_{t^*}, i) = 1$, yielding $V^i_{t^*}(\rho_{t^*}) = 1$ by (15). Now, $S_{t^*}(\rho_{t^*}) = V^1_{t^*}(\rho_{t^*}) + V^2_{t^*}(\rho_{t^*}) = 2$ at each ρ_{t^*}, and $(\rho_0, \rho_1, \ldots, \rho_{t^*})$ is independently distributed. As we showed in subsection 3.4, under these conditions, they will not reach an agreement before t^*. [Note that $y_{t^*} = 1$ and $y_t = 0$ thereafter, consistent with perpetual optimism.]

Example 2 also suggests the third and the most important rationale for requiring players' beliefs be firm: When a player is known to update his beliefs substantially with limited observation, his opponent may be willing to wait and let him observe, hoping that he will be convinced that she is right. For instance, in our example, players wait until t^*, when their opponents are expected to be convinced that they will no longer be able to make any offer.

To illustrate our rationale in a greater detail, consider the canonical case of affiliated ρ, where each player finds a player i more likely to be recognized at t if they know that she is recognized at some s. Assume that players are very optimistic about t and that they find being recognized at s as a strong evidence of being recognized at t. In that case, they must also be very optimistic about s. Now, player i thinks that, if they disagree until s, it is very likely that she will be recognized at s, and hence her opponent, namely j, will be convinced that i is very likely to be recognized at t, and will be willing to agree on i's terms. Being very *optimistic about convincing her opponent* by s, player i will not settle at the beginning unless she gets a high share. But her opponent is also very optimistic about convincing i by s, and thus may not be willing to give i what she wants for an agreement at the beginning. In fact, if i's continuation value at s increases substantially as she convinces j, and if s is close enough, then they would choose to wait until s – as we demonstrated in Example 2. Our Theorem assumes that the rate of learning is so slow that either the impact of convincing j is not so large, or the required time to convince him is too long to wait.

On the other hand, in canonical learning models, unless their beliefs are sufficiently firm, players update their beliefs substantially at the beginning, whence we have disagreement regimes at the beginning of the game, and thus a delay in reaching an agreement on the path of equilibrium. Analyzing such a canonical learning-model, in our next section we will show that agreement can be delayed, and will derive a strict bound on the settlement date.

6 A stylized model with learning

Without restricting how fast players update their beliefs, but assuming away the deadline effects, we will now explore when players reach an agreement in a canonical case, characterized by Assumption TU and Assumption XB below, which we will maintain throughout this section.

Under these two assumptions, we will show for $\bar{t} = \infty$ that there exists a predetermined date such that players will never reach an agreement before that date and will always agree thereafter. Throughout this section, we will take $T = \mathbb{N}$, i.e., $\bar{t} = \infty$. (Our results are also valid for any sufficiently large $\bar{t}$; see Yildiz, 2000).

Assumption XB Given any $t, s, r \in \mathbb{N}$ with $r \leq t \leq s$, and any history ρ_t^r of recognitions where player 1 is recognized r times out of t, for each $i \in N$, we have

$$P^i(\rho_s = 1 | \rho_t^r) = \frac{k_i + r + 1}{t + h + 2} \tag{24}$$

for some $k_i, h \in \mathbb{N}$ with $k_i \leq h$.

This assumption holds when each player believes that ρ is exchangeable, i.e., it is identically and independently distributed with some unknown parameter μ measuring the probability of $\{\rho_t = 1\}$, and μ is distributed with a beta distribution such that as if each i had a uniform distribution on $[0, 1]$ for μ and observed h (pre-bargaining) trials at an interim stage where player 1 was recognized k_i times (see Fudenberg and Levine, 1998). Of course, each player believes that his own h trials are relevant for this bargaining. We will write $K = k_1 - k_2$. While h measures the level of conviction in players' prior beliefs, $K/(h+2)$ will be shown to measure the initial level of optimism.

Note that, since i believes that ρ is identically distributed, his beliefs about ρ_s does not depend on s. Substituting (24) in definition of y, we obtain that

$$y_s(\rho_t^r) = \frac{k_1 - k_2}{t + h + 2} \equiv \frac{K}{t + h + 2} \equiv y_{(t)} \tag{25}$$

at each $\rho_t \in N^t$ and $s \geq t$. Two properties of y are important for us. Firstly, since players' beliefs about s do not depend on s, the discrepancy in beliefs about s does not depend on s, either. It depends only on the date the beliefs are held, i.e., $y_s(\rho_t) = y_t(\rho_t)$ for each $s \geq t$. Therefore, we denote it by $y_{(t)}$ – indexed only by the date the beliefs are held. Note that $y_{(t)}$ measures the discrepancy in players' beliefs about any recognition in future at the beginning of date t, while y_t denotes the entire process of the discrepancy in beliefs about the recognition at t, whose value at the beginning of t is $y_{(t)}$. When $k_1 \leq k_2$, we have $y_{(t)} \leq 0$ throughout so that we have agreement regimes throughout the game as in the Rubinstein-Stahl framework. We will assume that $k_1 > k_2$ (i.e., $K > 0$), rendering $y_{(t)} > 0$ throughout.

Second, y_t is *deterministic*, i.e., $y_t(\rho_s) = y_t(\rho_s')$ for each $\rho_s, \rho_s' \in N^s$ with $s \leq t$. This is due to our assumption that h is same for both players. This renders R and S also deterministic – even though V is *not* deterministic. (Recall that $S_t = V_t^1 + V_t^2$ and $R_t^i(\rho_t) = \max\{0, 1 - \delta S_{t+1}(\rho_t, i)\}$.)

Lemma 3 *Under Assumptions TU and XB, given any $t \in T = \mathbb{N}$ and $i \in N$, S_t and R_t^i are deterministic, and hence $R_t^1 = R_t^2 = R_t$ for some $R_t \in \mathbb{R}$.*

When S_{t+1} is deterministic, R_t^i is deterministic and $R_t^1 = R_t^2$. In that case, S_t will also be deterministic, giving us our Lemma. Since S_{t+1} and δ determine whether we have an agreement regime at $t \in T$, the event that players reach an agreement at t is also deterministic.

Writing

$$\Lambda_t = \sum_{s=t}^{\infty} \delta^{s-t} R_s \tag{26}$$

for the present value of all future rents, we express V and S in terms of Λ:

Lemma 4 *Assume TU and XB. Given any $\rho_t \in N^t$ and any $i \in N$, we have*

$$\begin{aligned} V_t^i(\rho_t) &= p_t^i(\rho_t)\,\Lambda_t & (27)\\ S_t(\rho_t) &= \left(1 + y_{(t)}\right)\Lambda_t. & (28) \end{aligned}$$

The first statement, (27), is a simple application of Proposition 2, Lemma 3, and the fact that $p_s^i(\rho_t) = p_t^i(\rho_t)$ for each $s \geq t$. Adding up (27) over the players, we obtain (28). Since p_t^i is stochastic, so is V_t^i. On the other hand, both $y_{(t)}$ and $\Lambda_t^{\bar{t}}$ are deterministic, and so is S_t.

Using (28), we can decouple (26), and obtain a difference equation for Λ. Consider any t with an agreement regime. Now, $R_t = 1 - \delta S_{t+1}$, hence (28) yields $R_t = 1 - \delta(1 + y_{(t+1)})\Lambda_{t+1}$. Moreover, by (26), we have $\Lambda_t = R_t + \delta\Lambda_{t+1}$. Combining these two equations, we obtain

$$\Lambda_t = 1 - \delta y_{(t+1)}\Lambda_{t+1}. \tag{29}$$

When we have a disagreement regime at t, we clearly have $\Lambda_t = \delta\Lambda_{t+1}$. These difference equations imply that Λ is uniformly bounded by 1.

By (28), we have an agreement regime at any $t-1 \in T$ iff

$$\Lambda_t \leq \frac{1}{\delta(1 + y_{(t)})} \equiv D(t). \tag{30}$$

On the right-hand side, as t gets larger, $y_{(t)}$ converges to zero so that $D(t)$ approaches to $1/\delta$. On the left-hand side, Λ is uniformly bounded by 1, which is less than $1/\delta$. Thus, inequality (30) holds for every sufficiently large t, yielding an agreement regime at $t-1$. For instance, whenever $t > t_0 \equiv \frac{\delta}{1-\delta}K - h - 2$, we have $y_{(t)} < \frac{1}{\delta} - 1$, yielding $D(t) > 1$, and thereby rendering an agreement regime at $t-1$.

Let us write

$$PA = \{t \in T | \Lambda_s \leq D(s)\ \forall s > t\}$$

for the interval of perpetual agreement, consisting of the dates t such that we have an agreement regime at each $s \geq t$. Since $t_0 \in PA$, $PA \subseteq \mathbb{N}$ is non-empty, and thus possesses some minimum $t^{**} \leq t_0$.

We will now derive a stricter upper-bound for t^{**}, the date the period of perpetual agreement starts. Writing

$$B(t) = \frac{1}{1+\delta y_{(t)}},$$

and using (29) repeatedly, we first obtain the following bounds for Λ_t.

Lemma 5 *Let $\bar{t} = \infty$, $\delta \in (0,1)$, and assume TU and XB. Then, $B(t) < \Lambda_t < B(t+1)$ at each $t \geq t^{**}$.*

We plot our bounds in Figure 2. $B(t)$ converges to 1 as $t \to \infty$. Hence, for sufficiently large t, $B(t)$ and $B(t+1)$ are arbitrarily close to each other, providing very precise bounds for Λ. As shown in the Figure, our bounds are valid only when $t \geq t^{**}$, because the recursive equation (29) holds only at agreement regimes.

Let us compare our bounds with D, the process that determines whether we have an agreement at any given date. Clearly, $B(t) < D(t)$ at each t so that our lower bound stays in agreement regime throughout. In comparing our upper bound $B(t+1)$ with $D(t)$, we write

$$\frac{1}{1+\delta y_{(t+1)}} \leq \frac{1}{\delta\left(1+y_{(t)}\right)} \iff y_{(t)} - y_{(t+1)} \leq \frac{1-\delta}{\delta}. \tag{31}$$

Since $y_{(t)} - y_{(t+1)} = \frac{K}{(t+h+2)(t+h+3)}$, this implies that $B(t+1) \leq D(t)$ iff

$$t \geq t_{cut} \equiv \frac{\sqrt{1+\frac{4\delta K}{1-\delta}}-1}{2} - h - 2. \tag{32}$$

Consequently, when $t \geq t_{cut}$, inequality $\Lambda_t \leq B(t+1) \leq D(t)$ holds, yielding an agreement regime at $t-1$.[8] This gives us our bound for t^{**}:

$$0 \leq t^{**} \leq \max\{0, \lfloor t_{cut} \rfloor\}. \tag{33}$$

We exhibit some values of t^{**} and t_{cut} for $h = K = 1$ at Table 1. When δ is sufficiently small,[9] we have agreement regimes throughout, in particular at the beginning of the game. As δ increases, some disagreement becomes possible; in fact, the perpetual agreement may be delayed for a long while.

It turns out that there cannot be any agreement regime before t^{**}, the date the period of perpetual agreement starts. This gives us our main Theorem in this section.

[8] Note that this is precisely the region where $y_{(t)} - y_{(t+1)} \leq (1-\delta)/\delta$, where we also have $y_{(s)} - y_{(s+1)} \leq (1-\delta)/\delta$ for each $s \geq t$. This is analogous to Theorem 2, which implies under NL and TU that we will have an agreement regime at $t-1$ whenever $y_s - y_{s+1} \leq (1-\delta)/\delta$ for each $s \geq t$.

[9] For instance, when $\delta < \left[\frac{4K}{(2h+7)^2-1}+1\right]^{-1}$, we have $\lfloor t_{cut} \rfloor \leq 0$ so that $t^{**} = 0$.

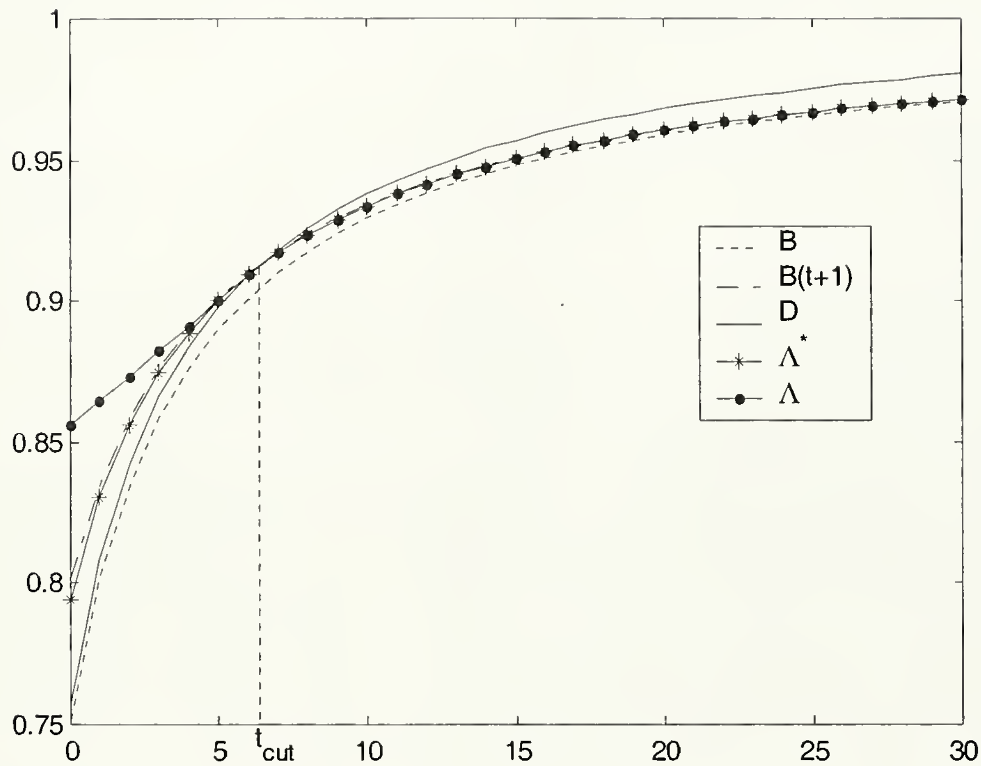

Figure 2: Functions D, B, $B(t+1)$, and Λ. Λ^* denotes the solution to (29), which is valid only on PA. Note that $t_{cut} = 6.4624$. $[\delta = 0.99,\ h = K = 1,\ \bar{t} = 500.]$

δ	$\lfloor t_{cut} \rfloor$	t^{**}	$\delta^{t^{**}}$
0.95	0	0	1
0.99	6	6	0.9415
0.999	28	27	0.9733
$1 - 10^{-6}$	996	996	0.9990
$1 - 10^{-9}$	31619	31618	1

Table 1: t_{cut} and t^{**} vs. δ. ($h = K = 1$, $\bar{t}$ is very large.)

Theorem 5 *Let $\bar{t} = \infty$, $\delta \in (0,1)$, and assume TU and XB. Then, there exists a $t^{**} \in T$ such that, at each $t \geq t^{**}$, players reach an agreement immediately if they have not reached an agreement yet, and they do not reach an agreement before t^{**}.*

Proof. By definition, we have an agreement regime at each $t \geq t^{**} \equiv \min PA$, hence we will only show that we have a disagreement regime at each $t < t^{**}$. If $t^{**} = 0$, this is vacuously true, so, we assume that $t^{**} > 0$. In that case, $t^{**} - 1 < t_{cut}$ by (33); and we have a disagreement regime at $t^{**} - 1$ by definition of t^{**}. Now we will show that, whenever we have a disagreement regime at any $t < t_{cut}$, we will have a disagreement regime at $t - 1$, showing by mathematical induction that we have a disagreement regime at each $s \leq t^{**} - 1$.

To this end, take some $t < t_{cut}$ with a disagreement regime so that $S_{t+1} > 1/\delta$. We have $R_t = 0$, and hence $\Lambda_t = \delta\Lambda_{t+1}$. By (28), this yields

$$\dot{S_t} = \left(1 + y_{(t)}\right)\delta\Lambda_{t+1} = \frac{\delta\left(1 + y_{(t)}\right)}{1 + y_{(t+1)}} S_{t+1}.$$

Hence, $S_t \geq S_{t+1}$ whenever $1 + y_{(t+1)} \leq \delta(1 + y_{(t)})$, i.e., whenever $y_{(t)} - \frac{1}{\delta}y_{(t+1)} \geq \frac{1-\delta}{\delta}$. But this is always true: $y_{(t+1)} \geq 0$ and $t < t_{cut}$, hence $y_{(t)} - \frac{1}{\delta}y_{(t+1)} \geq y_{(t)} - y_{(t+1)} \geq \frac{1-\delta}{\delta}$. Therefore, we have $S_t \geq S_{t+1} > 1/\delta$, and hence a disagreement regime at $t - 1$. ■

For a very canonical case, Theorem 5 states that, unless players' initial beliefs are so firm that our Agreement Theorems apply, reaching an agreement will be delayed for a while. For, typically, at the beginning of a learning process players are more open to new information, in the sense that they update their beliefs substantially as they observe which player gets a chance to make an offer. Knowing this, each player waits, believing that the events are very likely to proceed in such a way that his opponent will change his mind. As time passes, they become experienced. In this way, two things occur simultaneously, both facilitating agreement. Firstly, having similar experiences, the discrepancy in their beliefs diminishes. Secondly, they become so closed minded that their opponents lose their hope to convince them and thus become more willing to agree in their terms. Therefore, after a while, they reach an agreement.

At the beginning of our game it is common knowledge that they will not reach an agreement until t^{**}, when they will reach an agreement no matter what happens by then. How they will share the pie at t^{**} will depend on how many times each player will have been recognized. Since they disagree about how many times each player is likely to be recognized by t^{**}, there is no consensus among our players on *how* they can better each of them by agreeing on a decision at the beginning. Therefore, they wait until t^{**} even though there *is* a consensus among them that there is an agreement at the beginning that would leave each player better off.

How long can they delay the agreement? We showed that the delay can be at most $t_{cut}(h, K, \delta)$. Now, given any K and δ, as $h \to \infty$, $t_{cut}(h, K, \delta) \to -\infty$, so that $t^{**} = 0$ for sufficiently large values of h. There are two factors behind this limit. Firstly, as $h \to \infty$,

players' initial beliefs become very firm, approximating an independently distributed ρ in the limit, thus by Theorem 1, they reach an agreement immediately. But more importantly, the discrepancy in beliefs also disappears since $y_{(t)} \leq y_{(0)} = K/(h+2)$, converging to 0 as $h \to \infty$. It would be more interesting then to set $K = y_o(h+2)$ for some $y_o \in (0,1]$ so that the initial level of optimism is always y_o. In that case, for $t > 0$, we will have $y_{(t)} = y_o/(\frac{t}{h+1}+1) \to y_o$ as $h \to \infty$. Now, $t_{cut} = \sqrt{ah+b} - h - 5/2$ where $a = y_o\delta/(1-\delta)$ and $b = 2a + 1/4$. As h increases, initially, t_{cut} also increases, possibly due to the increasing level of optimism. After a point, however, t_{cut} starts decreasing, and approaches to $-\infty$ in the limit. Once again, there will be no delay in agreement for large values of h, consistent with our Agreement Theorems. Finally, as $\delta \to 1$, $t_{cut}(h,K,\delta) \to \infty$. The intuition for this result is clear. Very patient players can wait very long to find out whose beliefs are more accurate.

We will now measure how long real-time delay we can have in the continuous-time limit. We measure index-time t in terms of real time by $\tau(t,n) = t/n$, where $n > 0$ measures the fineness of the grid. Now, discount rate is $\delta(n) = \exp(-r/n)$ where $r > 0$ is the real-time impatience. Clearly, $\exp(-r\tau(t,n)) = \delta^t$, and thus $\tau(t,n) = \frac{1}{r}\log(\delta^{-t})$. Therefore, the maximum delay in real time is $\frac{1}{r}\log\left(\delta^{-t_{cut}(h,K,\delta)}\right)$.

If h and K do not depend on n, we can measure the real-time delay in the continuous time limit by $\lim_{\delta\to 1}\frac{1}{r}\log\left(\delta^{-t_{cut}(h,K,\delta)}\right)$. One can check from (32) that, given any h and K, as $\delta \to 1$, $\delta^{-t_{cut}(h,K,\delta)} \to 1$ so that there is no real-time delay in the limit. That is, given any h and K, the delay in agreement can be bounded to be arbitrarily short by recognizing players sufficiently fast.

It is crucial for this result that h and K are taken to be independent of n so that the discrepancy in beliefs vanishes arbitrarily fast as n gets sufficiently large, i.e., the discrepancy at any real time τ is $K/(\tau n + h + 2)$, converging to 0 as n goes to ∞.

In updating his beliefs, a player might take into account how fast the players are recognized. In particular, one might attribute the recognition of a player at a given time to the player's innate abilities little if players are to be recognized very frequently. Therefore, in order to measure the effect of the frequent recognition purely, one may want to adjust h and K accordingly so that the discrepancy in beliefs at a given real-time does not depend on the fineness n of the grid much. To do this, let us take $h = n$ and $K = y_0(h+2)$ so that the discrepancy in beliefs at a given real time τ is $y_o/(\frac{n}{n+1}\tau + 1)$, which is approximately $y_o/(\tau+1)$ for large values of n. As $n \to \infty$, $\delta(n) \to 1$, and h and K go to ∞ in such a way that the initial level of optimism stays at y_o unchanged. We have already checked that, when these two limits are taken separately, in both cases $\delta^{t_{cut}}$ goes to 1, rendering immediate agreement in the limit. When $n \to \infty$, however, we compute that

$$\lim_{n\to\infty} \tau(t_{cut}(h(n), K(n), \delta(n)), n) = \sqrt{y_o/r} - 1$$

so that, in the continuous time limit, we can have very long real-time delays if the players are both patient (with low r) *and* optimistic.

7 Impact of Risk-Aversion

In this section, reinstating Assumption NL, we wish to demonstrate the impact of risk aversion on the problem of reaching an agreement. We present a case where players disagree at the beginning of the game even though $y \geq 0$ and the game can be arbitrarily long. That is, y^* of Theorem 3 needs to be strictly positive, and thus Theorem 1 could not be extended beyond TU in its full strength.

Example 3 *Assuming NL, take* $U = \{u \in [0,1]^2 | (u^1)^2 + (u^2)^2 \leq 1\}$,[10] $\bar{t} > 3$, $\delta \in (0.907, 1)$, *and*

$$\begin{aligned} p^1 &= (1, 1, 1/2, 0, 0, \ldots, 0) \\ p^2 &= (0, 1, 1/2, 1, 1, \ldots, 1), \end{aligned}$$

so that $y = (0, 1, 0, 0, \ldots, 0)$. *We will have a disagreement regime at* $t = 0$.

Now, by Proposition 2, $V_3 = (0, 1)$. *Hence, at* $t = 2$, *if recognized, players 1 and 2 will offer* $(\sqrt{1-\delta^2}, \delta)$ *and* $(0, 1)$, *respectively. The offers will be accepted. Thus,*

$$V_2 = \frac{1}{2}\left(\sqrt{1-\delta^2}, \delta\right) + \frac{1}{2}(0, 1) = \left(\frac{\sqrt{1-\delta^2}}{2}, \frac{1+\delta}{2}\right).$$

At the beginning of date $t = 1$, *player 1 is certain that he will be recognized, and will offer* $(\sqrt{1 - (\delta V_2^2)^2}, \delta V_2^2)$, *which will be accepted. Thus, his continuation value will be*

$$V_1^1 = \sqrt{1 - (\delta V_2^2)^2} = \sqrt{1 - \delta^2 \left(\frac{1+\delta}{2}\right)^2}. \tag{34}$$

Likewise,

$$V_1^2 = \sqrt{1 - (\delta V_2^1)^2} = \sqrt{1 - \delta^2 \frac{1-\delta^2}{4}}. \tag{35}$$

We have a disagreement regime at $t = 0$ *if* $(V_1^1)^2 + (V_1^2)^2 > 1/\delta^2$. *By (34) and (35), this is the case when* $2 - \frac{\delta^2}{2}(1+\delta) > \frac{1}{\delta^2}$. *Writing* $f(\delta) = 2 - \frac{\delta^2}{2}(1+\delta) - \frac{1}{\delta^2}$, *we observe that* $f(1) = 0$, $f'(1) = -1/2 < 0$, *thus* $f(\delta) > 0$ *whenever* δ *is sufficiently close to 1. In fact, one can check that, for any* $\delta \in (0.907, 1)$, *we have* $f(\delta) > 0$, *hence we have a disagreement regime at* $t = 0$.

[10] This would be the case if players were dividing a dollar and each had utility function $x \mapsto \sqrt{x}$.

In a game with perpetual optimism, each player adjusts his expectations down, knowing that his opponent has very optimistic beliefs – even though he believes that his opponent is wrong. With transferable utility this adjustment was enough for agreement. When players are risk-averse, however, uncertainty about which player will get the rent lowers their continuation values at the beginning of any date with an agreement regime, increasing weakly the rent for the player recognized at the previous date. If in addition players are highly optimistic for this date, such an increase may be high enough for preventing them from agreeing at the date before – as our Example demonstrates.

8 Conclusion

As suggested by Edgeworth (1881), with complete information, in equilibrium, bargaining results in an optimal outcome. Given that the delays are costly, this also implies that the agreement is reached immediately. Yet, the agreement is delayed as a rule in real life. A prominently proposed explanation for the delay is parties' excessive optimism.

In this paper, we analyzed the problem of reaching an agreement in a model where players are possibly optimistic about the recognition process, the ultimate source of bargaining power in sequential bargaining with no outside option. In order to see the pure impact of optimism, in our analysis we relaxed only the common prior assumption of complete information, and adhered to the equilibrium.

We showed that the excessive optimism alone cannot be a reason for the delay. When players are sufficiently optimistic for a long while, recognizing that his opponent will remain optimistic for a long while, each player will lower his expectations about the future so that they will reach an agreement immediately. In other words, presence of perpetual optimism includes also the bad news for each player that his opponent will remain optimistic, inducing a form of pessimism that moderates his optimism to the extent that they settle at the beginning.

Corroborating this, we further showed in a special case that we will have an immediate agreement so long as there are no sudden jumps in the level of optimism, providing a rationale for the immediate agreement result in complete-information models, where the level of optimism is constantly nil. It is not the absence or the presence of optimism that determines whether we have an immediate agreement; that is determined by the *change* in the level of optimism.

Relaxing the common prior assumption enriches our understanding further. Taking differences in information to be the only source of differences in beliefs, the common prior assumption allows only one notion of convincing, namely communicating the privately known information in a credible way. While the delays are used as a means to convince the other party in this sense, another notion of convincing becomes salient in our model, providing a distinct rationale for delay. When players' beliefs are substantially different from each other, a player may be

willing to wait so that his opponent will observe more facts and have a better understanding of the truth, which is presumably like our player thinks.

A Appendix – Monotonicity of payoffs with respect to beliefs

Here we will show for affiliated recognition processes that a player's equilibrium payoff increases as he becomes more likely to make an offer in the future. We will first present an example showing that this is not true when the recognition process is not affiliated.

Example 4 *Take* $U = \left\{(u^1,u^2) \in [0,1]^2 \,|u^1 + u^2 \leq 1\right\}$, $\bar{t} = \infty$, $\delta \in (3-\sqrt{5},1)$, *and consider the following family of recognition processes* ρ, *parametrized by player 1's beliefs about* ρ_1. *Under* P^1, ρ *is independently distributed with* $P^1(\rho_0 = 1) = 0$, $P^1(\rho_1 = 1) = \pi$, *and* $P^1(\rho_s = 1) = 1/2$ *at each* $s > 1$. *Player 2's beliefs are similar to those of player 1, except that player 2 believes that he will not be recognized at* $t = 1$, *and if he is recognized at* $t = 1$, *he will never be recognized again. That is, given any* $\boldsymbol{\rho}_t = (\rho_0(\omega), \rho_1(\omega), \ldots, \rho_{t-1}(\omega))$ *with* $t \geq 2$, *for any* $s \geq t$, *we have*

$$P^2(\rho_s = 2|\boldsymbol{\rho}_t) = \begin{cases} 1/2 & \text{if } \rho_1(\omega) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

and $P^2(\rho_0 = 2) = 1$ *and* $P^2(\rho_1 = 2|\rho_0) = 0$. *We will show that player 1's continuation value at the beginning of date 1 is decreasing with* $\pi \equiv P^1(\rho_1 = 1)$. *Hence, when* π *gets higher, player 2 will offer player 1 a lower share at date 0; and the offer is to be accepted.*

If player 1 is recognized at $t = 1$, *we will have the Rubinstein-Stahl framework with identical players, hence he will offer* $(1-\delta/2, \delta/2)$, *which will be accepted. (The continuation values at* $t = 2$ *are* $1/2$.) *Take any* $\boldsymbol{\rho}_t$ *where player 2 is recognized at date 1. Now, Player 2 will always think that he will never make an offer; hence, by Proposition 2,* $V_{t+1}^2(\boldsymbol{\rho}_{t+1}) = 0$. *Hence at* t, *if recognized, player 1 offers* $(1,0)$, *which is to be accepted. Thus, by (20),* $V_t^1(\boldsymbol{\rho}_t) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot \delta V_{t+1}^1(\boldsymbol{\rho}_{t+1})$. *Since* $V_t^1(\boldsymbol{\rho}_t) = V_{t+1}^1(\boldsymbol{\rho}_{t+1})$, *we then have* $V_t^1(\boldsymbol{\rho}_t) = 1/(2-\delta)$. *Therefore, if player 2 is recognized at date 1, he will offer* $(\delta/(2-\delta), 1-\delta/(2-\delta))$, *which is to be accepted. The continuation value of player 1 at the beginning of date 1 is then*

$$V_1^1 = \pi\cdot(1-\delta/2) + (1-\pi)\cdot\frac{\delta}{2-\delta}.$$

Since $\delta \in (3-\sqrt{5},1)$, $\delta/(2-\delta) > 1-\delta/2$, *hence* V_1^1 *is decreasing with* π. *(Note that* $V_1^2 = \delta/2$, *hence* $V_1^1 + V_1^2 \leq 1$, *thus they reach an agreement at* $t = 0$.)

Proposition 2 stated that being recognized at some t contains a good news for a player that he will extract some non-negative rent. In our example, for player 1, the recognition at date 1 comes also with the bad news that his opponent is not excessively pessimistic, hence now it is common knowledge that, if they wait, he will not be able to get more than half the dollar. This bad news is dominant; and being recognized at date 1 is overall bad news for player 1. If player 1 comes to believe that he is more likely to receive such bad news, he will have a weaker bargaining position at date 0, in which case he will get a smaller share in the dollar.

Now we focus on a canonical class of recognition processes, where all these news are good. A recognition-process ρ is said to be (weakly) *affiliated* iff

$$P^i\left(\rho_s = i|\rho_{t_0} = i, \rho_{t_1}, \rho_{t_2}, \ldots, \rho_{t_n}\right) \geq P^i\left(\rho_s = i|\rho_{t_0} \neq i, \rho_{t_1}, \rho_{t_2}, \ldots, \rho_{t_n}\right)$$

for each finite sequence $(t_0, t_1, \ldots, t_n)$ of non-negative integers, for each $i \in N$ and $s \in \mathbb{N}$. Note that, if ρ is independently distributed or exchangeable, it is weakly affiliated. Note also that we define the affiliation among all the members of a process, while it is usually defined between two random variables, say ρ_s and ρ_{t_0}, which corresponds to the sequence with $t_0 = t_1 = \ldots = t_n$. If a process ρ is affiliated, then we have

$$P^i\left(\rho_s = i|i, \boldsymbol{\rho}_t\right) \geq P^i\left(\rho_s = i|j, \boldsymbol{\rho}_t\right) \text{ and } P^j\left(\rho_s = j|i, \boldsymbol{\rho}_t\right) \leq P^j\left(\rho_s = j|j, \boldsymbol{\rho}_t\right) \tag{36}$$

at any two histories $(i, \boldsymbol{\rho}_t)$ and $(j, \boldsymbol{\rho}_t)$ that start with distinct players i and j at date 0, but continues with identical players.

Proposition 3 *Given any two games $G^{\bar{t}}[\delta, \rho]$ and $G^{\bar{t}}[\delta, \tilde{\rho}]$ with*

$$P^i\left(\tilde{\rho}_s = i|\boldsymbol{\rho}_t\right) \geq P^i\left(\rho_s = i|\boldsymbol{\rho}_t\right) \text{ and } P^j\left(\tilde{\rho}_s = j|\boldsymbol{\rho}_t\right) \leq P^j\left(\rho_s = j|\boldsymbol{\rho}_t\right) \tag{37}$$

at each s and $\boldsymbol{\rho}_t$ for some distinct players i and j, if ρ is affiliated, then

$$V^i[\tilde{\rho}] \geq V^i[\rho] \text{ and } V^j[\tilde{\rho}] \leq V^j[\rho], \tag{38}$$

where the terms in $[\cdot]$ indicates the underlying recognition process.

That is, when the recognition process is affiliated, if players' beliefs change in such a way that each player finds a player i at least as likely to make an offer as before, player i (if anything) gains and his opponent (if anything) loses. A similar proposition for the Rubinstein-Stahl framework can be found in Merlo and Wilson (1995), where the recognition process is affiliated by their modelling assumptions.

Proof. It suffices to prove the Proposition for finite-horizon case. For, the continuation values in an infinite-horizon game is defined as limits of the continuation values in finite-horizon games; and the inequalities will remain to hold as we take the limit.

Assuming that $\bar{t} \in \mathbb{N}$, we will use mathematical induction on the length $\bar{t}$. Firstly, for $\bar{t} = 0$, we simply have $V^i_{\bar{t}}[\tilde{\rho}] = V^i_{\bar{t}}[\rho] = V^j_{\bar{t}}[\tilde{\rho}] = V^j_{\bar{t}}[\rho] = 0$, and our Proposition is trivially satisfied. Assume that our Proposition holds for all games of some length $\hat{t}$ and for every $\left(\Omega, P^1, P^2\right)$. Setting $\bar{t} = \hat{t} + 1$, take any two games $G^{\bar{t}}[\delta, \rho]$ and $G^{\bar{t}}[\delta, \tilde{\rho}]$ satisfying the hypotheses of the Proposition. Given any history $\mathbf{h}_{1,0} = (k, \mathbf{u}_1, \text{Reject}) \in N \times U \times \{\text{Reject}\}$, consider subgames $G^{\bar{t}}_{\mathbf{h}_{1,0}}[\delta, \rho]$ and $G^{\bar{t}}_{\mathbf{h}_{1,0}}[\delta, \tilde{\rho}]$ starting at $\mathbf{h}_{1,0}$, which are of length $\hat{t} = \bar{t} - 1$. Conditional on that k is recognized at 0, these subgames trivially satisfy (37) and process $\left(\rho_1, \rho_2, \ldots, \rho_{\bar{t}-1}\right)$ is affiliated (for ρ is affiliated). Hence, by our induction assumption, (38) holds for subgames $G^{\bar{t}}_{\mathbf{h}_{1,0}}[\delta, \rho]$ and $G^{\bar{t}}_{\mathbf{h}_{1,0}}[\delta, \tilde{\rho}]$, i.e., we have

$$V^i_t[\tilde{\rho}](\boldsymbol{\rho}_t) \geq V^i_t[\rho](\boldsymbol{\rho}_t) \text{ and } V^j_t[\tilde{\rho}](\boldsymbol{\rho}_t) \leq V^j_t[\rho](\boldsymbol{\rho}_t) \tag{39}$$

at each $t \geq 1$ and each $\boldsymbol{\rho}_t$. Thus, we only need to show that $V^i_0[\tilde{\rho}] \geq V^i_0[\rho]$ and $V^j_0[\tilde{\rho}] \leq V^j_0[\rho]$. Recall that, by (15), we have $V^i_0[\tilde{\rho}] = P^i(\tilde{\rho}_0 = i)\, m^i(\delta V_1[\tilde{\rho}](i)) + P^i(\tilde{\rho}_0 = j)\, \delta V^i_1[\tilde{\rho}](j)$ and $V^i_0[\rho] =$

$P^i(\rho_0 = i)\, m^i(\delta V_1[\rho](i)) + P^i(\rho_0 = j)\, \delta V_1^i[\rho](j)$. Hence, we can decompose $V_0^i[\tilde{\rho}] - V_0^i[\rho]$ as

$$
\begin{aligned}
V_0^i[\tilde{\rho}] - V_0^i[\rho] &= P^i(\tilde{\rho}_0 = i)\left[m^i(\delta V_1[\tilde{\rho}](i)) - m^i(\delta V_1[\rho](i))\right] && (40)\\
&\quad + P^i(\tilde{\rho}_0 = j)\left[\delta V_1^i[\tilde{\rho}](j) - \delta V_1^i[\rho](j)\right] && (41)\\
&\quad + \left[P^i(\tilde{\rho}_0 = i) - P^i(\rho_0 = i)\right]\left[m^i(\delta V_1[\rho](i)) - \delta V_1^i[\rho](j)\right]. && (42)
\end{aligned}
$$

We will show that all the terms in $[\cdot]$'s are non-negative. Firstly, note that $m^i(v)$ is non-decreasing with v^i and non-increasing with v^j. Hence, by (39), $m^i(\delta V_1[\tilde{\rho}](i)) - m^i(\delta V_1[\rho](i)) \geq 0$. By (39), we also have $\delta V_1^i[\tilde{\rho}](j) - \delta V_1^i[\rho](j) \geq 0$; thus the terms in lines (40) and (41) are non-negative. To show that $m^i(\delta V_1[\rho](i)) - \delta V_1^i[\rho](j) \geq 0$, we need to compare the subgames $G_i^{\bar{t}}[\delta, \rho]$ and $G_j^{\bar{t}}[\delta, \rho]$ starting at histories (i, u, Reject) and (j, u, Reject), respectively. Isomorphic to each such subgame $G_k^{\bar{t}}[\delta, \rho]$, we have a game $G^{\bar{t}-1}\left[\delta, \rho^k\right]$ where

$$P^l\left(\rho_s^k = l | \boldsymbol{\rho}_t\right) = P^l\left(\rho_s = l | k, \boldsymbol{\rho}_t\right)$$

at each $\boldsymbol{\rho}_t$ and $l \in N$. (ρ and ρ^k can be defined on different spaces.) Since ρ is affiliated, by (36), we have $P^i\left(\rho_s^i = i | \boldsymbol{\rho}_t\right) \geq P^i\left(\rho_s^j = i | \boldsymbol{\rho}_t\right)$ and $P^j\left(\rho_s^i = j | \boldsymbol{\rho}_t\right) \leq P^j\left(\rho_s^j = j | \boldsymbol{\rho}_t\right)$ at each $\boldsymbol{\rho}_t$. Since ρ is affiliated, so are ρ^i and ρ^j. Therefore, by our induction assumption, we have $V^i\left[\rho^i\right] \geq V^i\left[\rho^j\right]$; in particular, $V_1^i[\rho](i) = V_0^i\left[\rho^i\right] \geq V_0^i\left[\rho^j\right] = V_1^i[\rho](j)$. Similarly, we have $V_1^j[\rho](i) \leq V_1^j[\rho](j)$. Since m^i is non-decreasing in v^i and non-increasing in v^j, it follows that $m^i(\delta V_1[\rho](i)) \geq m^i(\delta V_1[\rho](j)) \geq \delta V_1[\rho](j)$, where the last inequality is due to $m^i(v) \geq v$. Therefore, $m^i(\delta V_1[\rho](i)) - \delta V_1[\rho](j) \geq 0$, and thus the term in line (42) is non-negative, showing that $V_0^i[\tilde{\rho}] - V_0^i[\rho] \geq 0$, and thus $V^i[\tilde{\rho}] \geq V^i[\rho]$. Similarly, we have $V^j[\tilde{\rho}] \leq V^j[\rho]$; and by induction hypothesis, the proof is complete. ∎

B Appendix–Proofs

Here we will present the proofs that we omitted in the text. We will present these proofs under the titles of the sections they belong to. When it is needed, we will write the underlying parameters in [], e.g., $V[\bar{t}, p]$, $V[p]$, etc. We will also use the supremum metric for the functions, i.e., $|p - q| = \max_{t,i,\boldsymbol{\rho}_s}\left|p_t^i(\boldsymbol{\rho}_s) - q_t^i(\boldsymbol{\rho}_s)\right|$, and we write $p > \lambda \in \mathbb{R}$ iff $p_t^i(\boldsymbol{\rho}_s) > \lambda$ for each $(t, i, \boldsymbol{\rho}_s)$.

B.1 Agreement Theorem

Now we will prove Theorems 3 and 4. Towards this goal, we will first develop our notion of measuring the size of the pie. Then, we will show that, under NL, the lengths of disagreement periods are uniformly bounded. Then, we will show that our result holds with strict inequalities for the case when y is identically 1. Finally, invoking a continuity property of equilibrium payoffs with respect to the belief structures, we will obtain a stronger version of Theorem 3, which will also imply Theorem 4.

Measuring the size of the pie Define $\Sigma : \mathbb{R}_+^2 \to \mathbb{R}_+$, measuring the size of each payoff vector $v \geq 0$ relative to U, by setting

$$\Sigma(v) = \frac{\|v\|}{\max\{\lambda \|v\| \,|\, \lambda v \in U, \lambda \in \mathbb{R}_+\}}$$

at each $v \in \mathbb{R}^2_+ \backslash \{0\}$ and setting $\Sigma(0) = 0$, where $\|\cdot\|$ is any given norm on $\mathbb{R}^2$. Thus defined, Σ is continuous, increasing, and homogenous of degree 1, i.e., $\Sigma(\lambda v) = \lambda\Sigma(v)$ at each $\lambda \geq 0$ and $v \in \mathbb{R}^2_+$. Moreover, $U \cap \mathbb{R}^2_+ = \{v|\Sigma(v) \in [0,1]\}$; and the Pareto-frontier is defined by $\Sigma(v) = 1$; in particular, we have $\Sigma(m^1(v), v^2) = \Sigma(v^1, m^2(v)) = 1$. Given any t, we write $S_t = \Sigma(V_t)$ for the perceived size of the pie at the beginning of date t; we have an agreement regime at any (ρ_t, i) iff $S_{t+1}(\rho_t, i) \leq 1/\delta$. Given any v, we will write $m(v) = (m^1(v), m^2(v))$.

Disagreement under NL Under Assumption NL, the length of an interval of disagreement regimes is uniformly bounded. For, if we have a disagreement regime at any given t, we have $m(\delta V_{t+1}) = \delta V_{t+1}$, and hence by equation (15), we have $V_t = \delta V_{t+1}$, yielding $S_t = \Sigma(V_t) = \Sigma(\delta V_{t+1}) = \delta\Sigma(V_{t+1}) = \delta S_{t+1}$. This holds at each t with a disagreement regime; we must thus have $S_t = \delta^{\hat{t}-t} S_{\hat{t}}$ at each such t, where $\hat{t}$ is the date of the first agreement regime after t. Hence, the length of such an interval of disagreement regimes is

$$L(S_{\hat{t}}, \delta) = \left\lceil \frac{\log(S_{\hat{t}})}{\log(1/\delta)} \right\rceil - 1 \leq \left\lceil \frac{\log(\Sigma(m(0)))}{\log(1/\delta)} \right\rceil - 1 \equiv \bar{L}(\delta),$$

where the inequality is due to the fact that $S_{\hat{t}} \leq \Sigma(m(0))$.

Extreme case Writing $\mathbf{1}$ for the belief structure p with $p^i_t = 1$ at each t and i, we now analyze the equilibrium of any finite-horizon game $G^{\bar{t}}[\delta, \mathbf{1}]$. ($y$ is also identically 1.)

Consider the end of the game. Since $V_{\bar{t}} = 0$, by (15), we have $V_{\bar{t}-1} = m(0)$, and hence $S_{\bar{t}-1} = \Sigma(m(0)) \geq 1$. When $S_{\bar{t}-1} \in [1, 1/\delta]$, we have an agreement regime at $\bar{t}-2$. If $S_{\bar{t}-1} > 1/\delta$, then we have disagreement regimes throughout the interval $\{\bar{t} - \bar{L}(\delta) - 1, \ldots, \bar{t} - 2\}$; and we have an agreement regime at $\bar{t} - \bar{L}(\delta) - 2$, where $\bar{L}(\delta) = L(S_{\bar{t}-1}, \delta)$ as $S_{\bar{t}-1} = \Sigma(m(0))$. By construction, $S_{\bar{t}-\bar{L}(\delta)-1} \in [1, 1/\delta]$. Together with our next lemma, this will guarantee that we have an agreement regime at each $t \leq \bar{t} - \bar{L}(\delta) - 2$.

Lemma 6 *Take any finite-horizon game $G^{\bar{t}}[\delta, \mathbf{1}]$. Given any $t \in T$, if $S_{t+1} \in [1, 1/\delta]$, then $S_t \in [1, 2-\delta] \subset [1, 1/\delta)$.*

Proof. Since $S_{t+1} \leq 1/\delta$, we have agreement regime at t; and thus, by (15), $V_t = m(\delta V_{t+1})$. Now, $\Sigma(\delta V_{t+1}) = \delta\Sigma(V_{t+1}) = \delta S_{t+1} \in [\delta, 1]$; therefore, it suffices to show that $\Sigma(m(v)) \in [1, 2-\delta]$ whenever $\Sigma(v) \in [\delta, 1]$. Given any $v \in \mathbb{R}^2_+$, if $\Sigma(v) = 1$, then $m(v) = v$, hence $\Sigma(m(v)) = \Sigma(v) = 1 \in [1, 2-\delta]$.

Now take any $v \in \mathbb{R}^2_+$ with $\delta \leq \Sigma(v) < 1$. By definition, $m(v) > v$, hence $\Sigma(m(v)) > \Sigma(v) \geq 1$. Hence, we only need to show that $\Sigma(m(v)) \leq 2-\delta$. We accomplish this using a geometric construction. See Figure 3 for our notation and graphical exposition. In addition, we write $CONE$ for the convex hull of r_1 and r_2. [Note that v, w_1, $m(v)$, $w_2 \in CONE$.] We will show that $m(v) \leq C$. Since $m(v) \in CONE$ and Σ is increasing and convex, this implies $\Sigma(m(v)) \leq \max\{\Sigma((2-\delta)w_1), \Sigma((2-\delta)w_2)\} = 2-\delta$, completing the proof.

To prove that $m(v) \leq C$, we first note that, since $v < m(v)$, line $\tilde{l}$ has a positive slop so that A, B, C, v, and $m(v)$ are all linearly ordered. In this order, clearly, we have $A < B < C$. Moreover, B is the center of the rectangle defined by v, w_1, $m(v)$, and w_2, i.e., $B = v/2 + m(v)/2 = w_1/2 + w_2/2$. Hence, we have $v < B < m(v)$. Furthermore, we have $A \leq v$. [For otherwise, we would have $v < A < B$

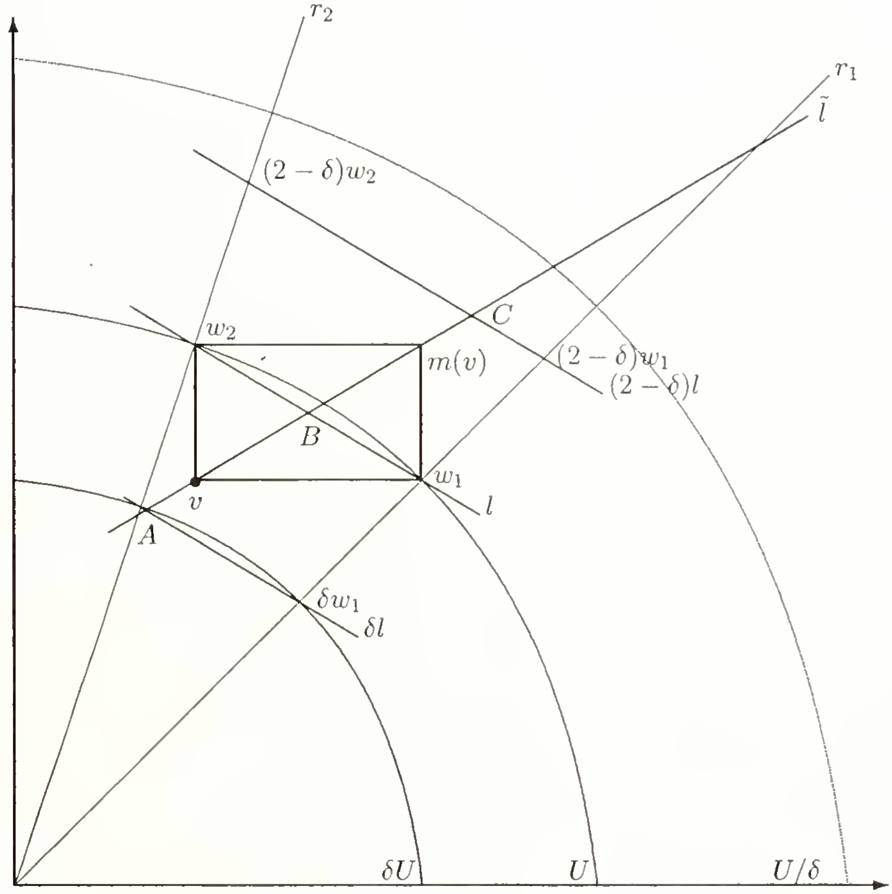

Figure 3: An illustration for Lemma 6.

so that $A \in CONE$ and hence $A = \lambda\delta w_1 + (1-\lambda)\delta w_2$ for some $\lambda \in [0,1]$. Since δU is convex, we would then have $A \in \delta U$, i.e., $\Sigma(A) \leq \delta$. But $v < A$ also implies that $\delta \leq \Sigma(v) < \Sigma(A)$; we have a contradiction.] On the other hand, since lines l, δl, and $(2-\delta)\, l$ are parallel to each other, we have

$$\frac{\|B-A\|}{\|C-B\|} = \frac{\|w_1 - \delta w_1\|}{\|(2-\delta)\, w_1 - w_1\|} = \frac{|1-\delta|\, \|w_1\|}{|2-\delta-1|\, \|w_1\|} = 1. \tag{43}$$

Hence,

$$\|m(v) - B\| = \|B - v\| \leq \|B - A\| = \|C - B\|,$$

where the first equality is due to $B = v/2 + m(v)/2$, the second one is due to $B > v \geq A$, and the last one is due to (43). Since $C > B$ and since $m(v)$ and C are ordered, this implies that $m(v) \leq C$. ■

Since $S_{\bar{t}-\bar{L}(\delta)-1} \in [1, 1/\delta]$, together with mathematical induction, Lemma 6 gives us the following Lemma.

Lemma 7 *Given any $G^{\bar{t}}[\delta, \mathbf{1}]$ with $\bar{t} < \infty$, at each $t \leq \bar{t} - \bar{L}(\delta) - 3$, we have $S_{t+1} \in [1, 2-\delta] \subset [1, 1/\delta)$.*

This is Theorem 3 for the extreme case of $p = \mathbf{1}$ with strict inequalities. On the other hand, continuation values are continuous with respect to the probability assessments p, as our next Lemma states. Since we can also bound p to be close to $\mathbf{1}$ by bounding y from below, this gives us Theorem 3, again with strict inequalities.

Lemma 8 *Given any $\delta \in (0,1)$, any $\bar{t} \in \mathbb{N}$, $t \in T$, and any $\epsilon > 0$, there exists some $\lambda > 0$ such that $|V_t[p](\boldsymbol{\rho}_t) - V_t[q](\boldsymbol{\rho}_t)| < \epsilon$ whenever $|p - q| \leq \lambda$.*

Proof. We use mathematical induction. Clearly, $V_{\bar{t}}[p] \equiv 0$ is continuous in p. Take any $t \in T$ and $\boldsymbol{\rho}_t$. By definition, we have $V_t^i[p](\boldsymbol{\rho}_t) = p_t^i(\boldsymbol{\rho}_t)\, m^i(\delta V_{t+1}[p](\boldsymbol{\rho}_t, i)) + (1 - p_t^i(\boldsymbol{\rho}_t))\, \delta V_{t+1}^i[p](\boldsymbol{\rho}_t, j)$ at each belief structure p, and each $i \in N$. Assume that both $p \mapsto V_{t+1}[p](\boldsymbol{\rho}_t, 1)$ and $p \mapsto V_{t+1}[p](\boldsymbol{\rho}_t, 2)$ are continuous. Since $m^i : \mathbb{R}_+^2 \to \mathbb{R}$ and the projection mappings $p \mapsto p_t^i(\boldsymbol{\rho}_t)$ and $v \mapsto v^i$ are all continuous, $p \mapsto V_t[p](\boldsymbol{\rho}_t)$ must be continuous as well. By induction hypothesis this completes the proof. ■

Lemmas 7 and 8 gives us Theorem 3 with strict inequalities:

Theorem 6 *Given any $\delta \in (0,1)$ and any finite $\bar{t} > 2$, there exists some $y^* < 1$ such that, for every game $G^{\bar{t}}[\delta, p]$ with $y \geq y^*$, we have $S_{t+1}(\boldsymbol{\rho}_{t+1}) \leq (2 - \delta + 1/\delta)/2 < 1/\delta$ (and hence an agreement regime) at each date $t \leq \bar{t} - \bar{L}(\delta) - 3$ and each $\boldsymbol{\rho}_{t+1}$.*

Proof. If $\bar{t} - \bar{L}(\delta) - 3 < 0$, then our Theorem is vacuously true; so, assume that $\bar{t} - \bar{L}(\delta) - 3 \geq 0$. Define $T_{early} = \{t \in T | t \leq \bar{t} - \bar{L}(\delta) - 3\}$, which is finite and non-empty. Given any $t \in T_{early}$, by Lemma 7, $\Sigma(V_{t+1}[\mathbf{1}]) \leq 2 - \delta < 1/\delta$; and since Σ is continuous, there exists some $\epsilon_t > 0$ such that $\Sigma(V_{t+1}[p](\boldsymbol{\rho}_{t+1})) \leq (2 - \delta + 1/\delta)/2$ whenever $|V_{t+1}[p](\boldsymbol{\rho}_{t+1}) - V_{t+1}[\mathbf{1}]| < \epsilon_t$. But by Lemma 8 there also exists some $\lambda_{\boldsymbol{\rho}_{t+1}} > 0$ such that $|V_{t+1}[p](\boldsymbol{\rho}_{t+1}) - V_{t+1}[\mathbf{1}]| < \epsilon_t$ whenever $|p - 1| \leq \lambda_{\boldsymbol{\rho}_{t+1}}$, i.e., $p_s^i(\boldsymbol{\rho}_s) \geq 1 - \lambda_{\boldsymbol{\rho}_{t+1}}$ for each $(s, \boldsymbol{\rho}_s, i)$. Set $\lambda = \min_{\boldsymbol{\rho}_{t+1}, t \in T_{early}} \lambda_{\boldsymbol{\rho}_{t+1}} > 0$ and $y^* = 1 - \lambda < 1$. Take any p with $y \geq y^*$. Now, given any $(s, \boldsymbol{\rho}_s, i)$, and any $\boldsymbol{\rho}_{t+1}$ with $t \in T_{early}$, we have $p_s^i(\boldsymbol{\rho}_s) \geq 1 - p_s^j(\boldsymbol{\rho}_s) + y_s(\boldsymbol{\rho}_s) \geq y_s(\boldsymbol{\rho}_s) \geq y^* = 1 - \lambda \geq 1 - \lambda_{\boldsymbol{\rho}_{t+1}}$, where $j \neq i$. Therefore, at each $\boldsymbol{\rho}_{t+1}$ with $t \in T_{early}$, we have $|V_{t+1}[p](\boldsymbol{\rho}_{t+1}) - V_{t+1}[\mathbf{1}]| < \epsilon_t$, yielding $\Sigma(V_{t+1}[p](\boldsymbol{\rho}_{t+1})) \leq (2 - \delta + 1/\delta)/2$. ■

Proof. (*Theorem 4*) Take any $\delta > 0$, and let $\epsilon_1 = 1/\delta - (2 - \delta + 1/\delta)/2 > 0$. Since Σ is continuous, by (21), for each $t \leq t^*$, there exists some $\hat{t}_t \in \mathbb{N}$ such that

$$|S_{t+1}[\infty, p] - S_{t+1}[\bar{t}, p]| \equiv |\Sigma(V_{t+1}[\infty, p]) - \Sigma(V_{t+1}[\bar{t}, p])| < \epsilon_1 \qquad (\forall p) \tag{44}$$

whenever $\bar{t} \geq \hat{t}_t$. This is true in particular for $\bar{t} \geq \hat{t} \equiv \max\{t^* + \bar{L}(\delta) + 3, \max_{t \leq t^*} \hat{t}_t\}$. By Theorem 6, there also exists some $y^* < 1$ such that, for every $G^{\hat{t}}[\delta, p]$ with $y \geq y^*$, we have $S_{t+1}[\hat{t}, p](\rho_{t+1}) \leq (2 - \delta + 1/\delta)/2 = 1/\delta - \epsilon_1$ at each ρ_{t+1} with $t \leq t^*$. In that case, by (44), we will have $S_{t+1}[\infty, p] \leq S_{t+1}[\hat{t}, p] + \epsilon_1 < 1/\delta$, i.e., an agreement regime at each $t \leq t^*$. ■

B.2 A stylized model with learning

Now we will present the proofs omitted in Section 6. The results in this section are stated for the infinite-horizon case. Here, we will first consider the finite-horizon case and then take the limit. We write $\bar{\mathbb{N}} = (\mathbb{N} \cup \{\infty\}) \setminus \{0\}$. We will first prove the following Lemma.

Lemma 9 *For any $\delta \in (0,1)$ and $\bar{t} \in \bar{\mathbb{N}}$, assume TU, XB and that $R_s^1[\bar{t}] = R_s^2[\bar{t}] = R_s[\bar{t}] \in \mathbb{R}$ for each $s \geq t$ for some $t < \bar{t}$. Also, write $\Lambda_t^{\bar{t}} = \sum_{s=t}^{\bar{t}-1} \delta^{s-t} R_s[\bar{t}]$. Then, given any $\rho_t \in N^t$ and any $i \in N$, $V_t^i[\bar{t}](\rho_t) = p_t^i(\rho_t)\Lambda_t^{\bar{t}}$ and $S_t[\bar{t}](\rho_t) = (1 + y_{(t)})\Lambda_t^{\bar{t}}$.*

Proof. Under our hypothesis, given any ρ_t and $s \geq t$, we have $E^i[1_{\{\rho_s = i\}} R_s^i[\bar{t}] | \rho_t] = R_s[\bar{t}] E^i[1_{\{\rho_s = i\}} | \rho_t] = R_s[\bar{t}] P^i(\rho_s = i | \rho_t) = R_s[\bar{t}] P^i(\rho_t = i | \rho_t) = p_t^i(\rho_t) R_s[\bar{t}]$, where the penultimate equality is due to the fact that player's beliefs about different dates are identical. Hence, by Proposition 2, we have $V_t^i[\bar{t}](\rho_t) = \sum_{s=t}^{\bar{t}-1} \delta^{s-t} E^i[1_{\{\rho_s = i\}} R_s^i[\bar{t}] | \rho_t] = p_t^i(\rho_t)\Lambda_t^{\bar{t}}$. This also implies that $S_t[\bar{t}](\rho_t) = V_t^1[\bar{t}](\rho_t) + V_t^2[\bar{t}](\rho_t) = p_t^1(\rho_t)\Lambda_t^{\bar{t}} + p_t^2(\rho_t)\Lambda_t^{\bar{t}} = (p_t^1(\rho_t) + p_t^2(\rho_t))\Lambda_t^{\bar{t}} = (1 + y_{(t)})\Lambda_t^{\bar{t}}$. ■

Lemmas 3 and 9 imply Lemma 4 as an immediate corollary. We now prove Lemma 3.

Proof. (*Lemma 3*) We will prove the lemma for any $\bar{t} \in \bar{\mathbb{N}}$. We first consider the case $\bar{t} \in \mathbb{N}$ and use mathematical induction: Firstly, $R_{\bar{t}-1}^1[\bar{t}] = R_{\bar{t}-1}^2[\bar{t}] = 1$, which is deterministic. Assume that $R_s^1[\bar{t}] = R_s^2[\bar{t}] = R_s[\bar{t}] \in \mathbb{R}$ for each $s \geq t+1$ for some $t < \bar{t}$. Then, by Lemma 9, $S_{t+1}[\bar{t}] = (1 + y_{(t+1)})\Lambda_{t+1}^{\bar{t}} \in \mathbb{R}$ and hence $R_t^1[\bar{t}] = R_t^2[\bar{t}] = \max\{0, 1 - \delta S_{t+1}[\bar{t}]\} \in \mathbb{R}$, showing by mathematical induction that $R_t^1[\bar{t}] = R_t^2[\bar{t}] = R_t[\bar{t}]$ for some $R_t[\bar{t}] \in \mathbb{R}$ for each $t < \bar{t}$. In that case, by Lemma 9, $S_t[\bar{t}] = (1 + y_{(t)})\Lambda_t^{\bar{t}} \in \mathbb{R}$ for each $t < \bar{t}$, proving our Lemma for finite-horizon case. In the infinite-horizon case, as $\bar{t} \to \infty$, we will show that $V_t[\bar{t}] \to V_t[\infty]$, and hence $S_t[\bar{t}] \to S_t[\infty]$ and $R_t[\bar{t}] \to R_t[\infty]$ at each t, showing that $S_t[\infty]$ and $R_t[\infty]$ are also deterministic. ■

To see that $V_t[\bar{t}] \to V_t[\infty]$, first observe that $\Lambda_t^{\bar{t}} = 1 - \delta y_{(t+1)}\Lambda_{t+1}^{\bar{t}}$ and $\Lambda_t^{\bar{t}} = \delta\Lambda_{t+1}^{\bar{t}}$ when we have agreement and disagreement regimes at t. Using the fact that $|\delta y_{(t+1)}| \leq \delta < 1$, one can check from these equalities that $\Lambda^{\bar{t}}$ is strongly stable backwards. Since $\Lambda_{\bar{t}-1}^{\bar{t}} = R_{\bar{t}-1}[\bar{t}] = 1$, $\Lambda^{\bar{t}}$ thus possesses a pointwise limit Λ^∞. Therefore, $V_t^i[\bar{t}](\rho_t) \to V_t^i[\infty](\rho_t) \equiv p_t^i(\rho_t)\Lambda_t^\infty$ everywhere.

This also shows that $\bar{s}[\infty]$ is an equilibrium. From now on, we will take $\bar{t} = \infty$ and suppress $\bar{t}$ in our expressions.

Towards proving Lemma 5, let us define functions B, $\bar{B}$, and C by setting $B(t) = \frac{1}{1+\delta y_{(t)}}$, $\bar{B}(t) = B(t+1)$ and $C(t) = \frac{y_{(t-1)}}{y_{(t)}} \frac{1}{1+\delta y_{(t-1)}}$ at each $t \in \mathbb{N}$. Note that $B(t) < \bar{B}(t) < C(t)$ at each t. For the last inequality, one can check that

$$C(t) - \bar{B}(t) = \frac{y_{(t-1)}(1 - \delta y_{(t+1)})}{K(1 + \delta y_{(t-1)})(1 + \delta y_{(t+1)})} > 0.$$

We start with the following Lemma:

Lemma 10 *Given any $t \in PA$, and $b, c \in \mathbb{R}$, we have*

$$\begin{aligned} \Lambda_{t+1} &= B(t+1) - b \iff \Lambda_t = \bar{B}(t) + \delta y_{(t+1)} b, &(45)\\ \Lambda_{t+1} &= C(t+1) + c \iff \Lambda_t = B(t) - \delta y_{(t+1)} c. &(46)\end{aligned}$$

Proof. Since $\Lambda_t = 1 - \delta y_{(t+1)} \Lambda_{t+1}$, we have

$$\Lambda_{t+1} = \frac{1}{1+\delta y_{(t+1)}} - b \iff \Lambda_t = 1 - \delta y_{(t+1)} \left[\frac{1}{1+\delta y_{(t+1)}} - b\right] = \frac{1}{1+\delta y_{(t+1)}} + \delta y_{(t+1)} b,$$

showing (45). Likewise,

$$\Lambda_{t+1} = \frac{y_{(t)}}{y_{(t+1)}} \frac{1}{1+\delta y_{(t)}} + c \iff \Lambda_t = 1 - \delta y_{(t+1)} \frac{y_{(t)}}{y_{(t+1)}} \frac{1}{1+\delta y_{(t)}} - \delta y_{(t+1)} c = \frac{1}{1+\delta y_{(t)}} - \delta y_{(t+1)} c,$$

showing (46). ■

Note that, whenever $\Lambda_{t+1} > B(t+1)$, we have $\Lambda_{t+1} = B(t+1) - b$ for some $b < 0$, and hence we have $\Lambda_t = \bar{B}(t) + \delta y_{(t+1)} b < \bar{B}(t)$. Likewise, whenever $\Lambda_{t+1} < C(t+1)$, we have $\Lambda_{t+1} = C(t+1) + c$ for some $c < 0$, and hence we have $\Lambda_t = B(t) - \delta y_{(t+1)} c > B(t)$. This gives us the following Lemma.

Lemma 11 *Given any $t \in PA$, if $B(t+1) < \Lambda_{t+1} < C(t+1)$, then $B(t) < \Lambda_t < \bar{B}(t)$.*

Lemma 12 *For any $t \in PA$, $B(t) < \Lambda_t < C(t)$.*

Proof. Take any $t \in PA$, and write

$$\theta_t^s = \begin{cases} 1 & \text{if } t = s \\ \prod_{j=t+1}^{s} \left(-\delta y_{(j)}\right) & \text{if } t < s \end{cases}$$

for each $s \geq t$. Then, using Lemma 10, we compute that

$$\Lambda_t = C(t) + \theta_t^{t+2l} \left[\Lambda_{t+2l} - C(t+2l)\right] - \sum_{0 \leq k \leq l-1} \theta_t^{t+2k} \left[C(t+2k) - \bar{B}(t+2k)\right]. \tag{47}$$

for each $t \in PA$, and $l \geq 0$. Using Lemma 10 and mathematical induction on l, one can easily check that (47) holds.

Equation (47) implies that $\Lambda_t < C(t)$ when l is sufficiently large. To see this, note first that, since $\left|-\delta y_{(t)}\right| < 1$, as $l \to \infty$, $\theta_t^{t+2l} \to 0$. Since we further have $|\Lambda_{t+2l} - C(t+2l)| < 1$ at each t, l, it follows that, as $l \to \infty$, $\theta_t^{t+2l} \left[\Lambda_{t+2l} - C(t+2l)\right] \to 0$. Second, $\theta_t^{t+2k} > 0$ for each k, as it consists of multiplication of evenly many negative numbers. Since $C(t+2k) - \bar{B}(t+2k)$ is always positive, it follows that $\sum_{0 \leq k \leq l-1} \theta_t^{t+2k} \left[C(t+2k) - \bar{B}(t+2k)\right]$ is positive, increasing in l, and hence bounded away from zero. Therefore, there exists $l' \in \mathbb{N}$ such that

$$\theta_t^{t+2l} \left[\Lambda_{t+2l} - C(t+2l)\right] - \sum_{0 \leq k \leq l-1} \theta_t^{t+2k} \left[C(t+2k) - \bar{B}(t+2k)\right] < 0 \tag{48}$$

whenever $l \geq l'$.

On the other hand, using (47) at $t+1$ and (46), one can also obtain

$$\begin{aligned}\Lambda_t &= B(t) - \delta y_{(t+1)}\theta_{t+1}^{t+2l+1}\left[\Lambda_{t+2l+1} - C(t+2l+1)\right] \\ &\quad +\delta y_{(t+1)} \sum_{0\leq k\leq l-1} \theta_{t+1}^{t+1+2k}\left[C(t+1+2k) - \bar{B}(t+1+2k)\right]. \end{aligned} \tag{49}$$

Of course, by (48), there exists some $l'' \in \mathbb{N}$ such that

$$-\delta y_{(t+1)}\theta_{t+1}^{t+2l+1}\left[\Lambda_{t+2l+1} - C(t+2l+1)\right] + \delta y_{(t+1)} \sum_{0\leq k\leq l-1} \theta_{t+1}^{t+2k+1}\left[C(t+2k+1) - \bar{B}(t+2k+1)\right] > 0 \tag{50}$$

whenever $l \geq l''$, whence we have $\Lambda_t > B(t)$. Therefore, for any $l \geq \max\{l', l''\}$, inequalities (48) and (50) simultaneously hold, hence by (47) and (49), we have $B(t) < \Lambda_t < C(t)$. ■

We can now prove Lemma 5.

Proof. (*Lemma 5*) When $t \notin PA$, our Lemma is vacuously true. Take any $t \in PA$. By Lemma 12, we have $B(t+1) < \Lambda_{t+1} < C(t+1)$, hence by Lemma 11, we have $B(t) < \Lambda_t < \bar{B}(t)$. ■

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